IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 7 Issue 7, July 2020
ISSN (Online) 2348 – 7968 | Impact Factor (2020) – 6.72
www.ijiset.com

The Stieltjes Moment Problem Related to the Hamberger Problem

G. K. Sanap
Head, Department of Mathematics, Sunderrao Solanke Mahavidyalaya, Majalgaon, Dist Beed (MS) India

Abstract :- This present paper deals with the Stieltjes moment problem related to the hamberger problem. A significant and required condition that there should exist an increasing function $\alpha(t)$ such as-

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...),$$

equations should have an increasing solution $\alpha(t)$ with infinitely many points of non-decrease and equations should have an increasing solution $\alpha(t)$ with a finite number of points of non-decrease .

Keywords :- Stieltjes Moment Problem, Hamberger problem, Hausdroff's moment problem, Increasing function ,Distribution function.

1 Introduction :- The moment problem on [0,1) is referred to as the Hausdroff's moment problem and the moment problem on R is called the Hamburger moment problem and thus $[0,\infty)$ is called the Stieltjes moment problem[1]. The Stieltjes Moment Problem seeks for a non decreasing positive distribution function on the semi-axis so that its moment match a given infinite sequence of positive real numbers [2-4].

Hausdroff``s moment problem often known as moment problem is given as below-

$$\{\mu_n\}_0^\infty : \mu_0, \mu_1, \mu_2, ...; \qquad (1,1)$$

There are other forms of existence to predict function $\alpha(t)$ of bounded variation in interval (0,1) for example -

$$\mu_n = \int_0^1 t^n d\alpha(t) \qquad (n = 0, 1, 2,...) \qquad (1,2)$$

Hence, this is corollary known as moment sequence. But, it has been noticed that every sequence (1,1) has them form (1,2) since (1,2) implies that-

$$|\mu_n| \leqq V[\alpha(t)]_0^1$$

the quantity on the right being the variation of $\alpha(t)$ on the interval (0, 1). Thus, in this way its sequence is confined. It was F. Hausdorff [1921a] to determine essential situation of sequence to be moment sequence. But, given that representation (1,2) if $\alpha(t)$ is a normalized function of corollary differences.

$$\alpha(0)=0, \quad \alpha(t)=\frac{\alpha(t+)+\alpha(t-)}{2} \qquad (0 < t < 1).$$

As normalization of the function $\alpha(t)$ does not varies the value of the integral (1,2) so, we can take over the loss with any disturbance to general prospect $\alpha(t)$ which is normalized. Infact, the entire discussion throw light upon it without any obstacle.

Equations (1,2) may be admired as an inter-changeability of the function $\alpha(t)$ into the sequence $\{\mu_n\}$. This inter-changeability is closely attached to the Laplace transform, is in fact the discrete analogue of the latter. For, if we substitute the integer n by the variable s in (1,2) and then make the change of variable $t = e^{-u}$, we obtain-

$$\mu_s = \int_0^\infty e^{-su} d[-\alpha(e^{-u})].$$

Hence, it is perfectly observed that enough importance relays on Laplace transform by a solution of the moment problem.

2 Material and Methods

The Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval. R.P. Boas [1939] has observed that in this case there is hardly any problem, as every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution[2].

We provide the proof of Boas. which will of course be enough to initialize the Stieltjes case.

The equations

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...)$$

For ever have a solution $\alpha(t)$ of closely related variation for which

$$\int_0^\infty |d\alpha(t)| < \infty .$$

We formed the other two sequences $\{\lambda_n\}_0^\infty, \{\nu_n\}_0^\infty$, such as;

$$\mu_n = \lambda_n - \nu_n$$

$$\lambda_n = \int_0^\infty t^n d\beta(t)$$

$$\nu_n = \int_0^\infty t^n d\gamma(t) \qquad (n = 0, 1, 2, ...),$$

3 Result and discussion

3.1 The Stieltjes Moment Problem Related to the Humberger Problem

The Stieltjes problem can also be applied as a special case of the Hamburger problem[7,16].

Axiom 3.1(a) A significant and required condition that there should exist an increasing function $\alpha(t)$ such as-

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...), \qquad (3.1,1)$$

The integrals are all converging, thus; the sequences $\{\mu_n\}_0^\infty$ and $\{\mu_n\}_1^\infty$ should be non-negative, or that the quadratic forms

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j \qquad (n = 0, 1, 2, ...) \qquad (3.1,2)$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j+1}\xi_i\xi_j \qquad (n = 0, 1, 2, ...) \qquad (3.1,3)$$

should be non-negative (definite or semi-definite).

According to Axiom - The equivalence of the two forms of the condition is apparent. We prove the result in the latter form involving quadratic forms. For the requirement, Let the sequence $\{\mu_n\}_0^\infty$ have the form (3.1,1). since we may regard $\alpha(t)$ as constant in the interval $(-\infty, 0)$. Proceeding further,

$$\mu_{n+1} = \int_0^\infty t^n d\beta(t) \qquad (n = 0, 1, ...)$$

$$\beta(t) = \int_0^t u d\alpha(u) \qquad (t \geqq 0).$$

As $\beta(t)$ is also an increasing so, Axiom(a) A necessary and required condition that there should be an existence of at least one increasing function $\alpha(t)$ in a way as

$$\mu_n = \int_{-\infty}^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...),$$

all the integrals converging, is that the sequence $\{\mu_n\}_0^\infty$ should be non-negative. And Axiom(b)A significantly required and necessary condition that the sequence $\{\mu_n\}_0^\infty$ should be non-negative definite (semi-definite) is that the quadratic forms

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j \qquad (n = 0, 1, 2, ...)$$

could be non-negative definite (semi-definite).

Axiom (a) and (b) show that the forms (3.1,3) are also non-negative.

Contradictory, suppose the forms (3.1,2) and (3.1,3) be non-negative and then considering the new sequence $\{\nu_n\}_0^\infty$ where,

$$\nu_{2n} = \mu_n \qquad (n = 0, 1, ...)$$

$$\nu_{2n+1} = 0 \qquad (n = 0, 1, ...).$$

If n = odd,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\nu_{i+j}\xi_i\xi_j = \sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}}\mu_{i+j}\xi_{2i}\xi_{2j}\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}}\mu_{i+j+1}\xi_{2i+1}\xi_{2j+1},$$

and now if n = even;

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\nu_{i+j}\xi_i\xi_j = \sum_{i=0}^{\frac{n}{2}}\sum_{j=0}^{\frac{n}{2}}\mu_{i+j}\xi_{2i}\xi_{2j}\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{n}{2}-1}\mu_{i+j+1}\xi_{2i+1}\xi_{2j+1},$$

This proves that the sequence $\{\nu_n\}_0^\infty$ is non-negative, and thus by Axiom (a) it stated that there lies an existence of an increasing function $\beta(t)$ in such a way-

$$\nu_n = \int_{-\infty}^{\infty} t^n d\beta(t) \qquad (n = 0, 1, 2, ...),$$

Otherwise,

$$\mu_n = \int_{-\infty}^{\infty} t^{2n} d\beta(t) \qquad (n = 0, 1, 2, ...)$$

$$0 = \int_{-\infty}^{\infty} t^{2n+1} d\beta(t) \qquad (n = 0, 1, 2, ...). \qquad (3.1,4)$$

Set

$$\gamma(t) = \frac{\beta(t) - \beta(-t)}{2} \qquad (-\infty < t < \infty).$$

Here, the function is odd which also satisfy the equations (3.1,4). It is increasing.

Set $\alpha(t) = 2\gamma(t^{1/2})(t \geqq 0)$. So, Then-

$$\mu_n = \int_{-\infty}^{\infty} t^{2n} d\gamma(t) = \int_0^{\infty} t^{2n} d\gamma(t) - \int_0^{\infty} t^{2n} d\gamma(-t)$$

by an obvious change of variables. But since $\gamma(t)$ is odd, this gives

$$\mu_n = 2\int_0^{\infty} t^{2n} d\gamma(t) \qquad (n = 0, 1, 2, ...)$$

$$= \int_0^{\infty} t^n d\left[2\gamma\left(t^{1/2}\right)\right]$$

$$= \int_0^{\infty} t^n d\alpha(t).$$

Since, $\alpha(t)$ is increasing in the interval $0 \leqq t < \infty$, we reached the expected outcome.

Clearly stated that we also have the following results.

Axiom 3.1(b) A significant and required condition that equations (3.1,1) should have an increasing solution $\alpha(t)$ with infinitely many points of non-decrease is that the forms (3.1,2) and (3.1,3) should all be non-negative definite or that the zero must be

$$\mu_0, \quad \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix}, \quad \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix}, \ldots \qquad (3.1,5)$$

$$\mu_1, \quad \begin{vmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{vmatrix}, \quad \begin{vmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 & \mu_5 \end{vmatrix}, \ldots$$

lesser than all the determinants.

Axiom 3.1(c) A required and significant condition that equations (3.1,1) should have an increasing solution $\alpha(t)$ with a finite number of points of non-decrease is that the forms (3.1,2) and (3.1,3) should all be non-negative & at least one of them being non-negative semi-definite.

We observe that it is not enough that the determinants (3.1,4) should be all positive.

3.2 The Humburger Problem-

According to the Hausdorff problem one may expect that it could be expectable to cope with the Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval. R.P. Boas [1939] has observed that in this case there is hardly any problem, as every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution [13].

We provide the proof of Boas. which will of course be enough to initialize the Stieltjes case[2-6].

Axiom 3.2(a) The equations

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...)$$

For ever have a solution $\alpha(t)$ of closely related variation for which

$$\int_0^\infty |d\alpha(t)| < \infty .$$

We formed the other two sequences $\{\lambda_n\}_0^\infty, \{\nu_n\}_0^\infty$, such as;

$$\mu_n = \lambda_n - \nu_n \qquad (3.2, 6)$$

$$\lambda_n = \int_0^\infty t^n d\beta(t) \qquad (3.2, 7)$$

$$\nu_n = \int_0^\infty t^n d\gamma(t) \qquad (n = 0, 1, 2, ...), \qquad (6.2, 8)$$

Now, here $\beta(t)$ and $\gamma(t)$ are closely related increasing functions. First choose $\lambda_0, \lambda_1, \nu_0, \nu_1$ as any non-negative numeric satisfying (3.2, 6). Proceeding by induction, Let, λ_k, ν_k for k = 0, 1, 2, ...,2n – 1 so that (3.2,6) holds and so that the determinants

$$[\lambda_0, \lambda_1, ..., \lambda_{2k}] = \begin{vmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_k \\ \lambda_1 & \lambda_2 & \dots & \lambda_{k+1} \\ . & . & \dots & . \\ \lambda_k & \lambda_{k+1} & \dots & \lambda_{2k} \end{vmatrix} \quad (3.2, 9)$$

$$[\lambda_1, \lambda_2, ..., \lambda_{2k+1}] = \begin{vmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{k+1} \\ \lambda_2 & \lambda_3 & \dots & \lambda_{k+2} \\ . & . & \dots & . \\ \lambda_{k+1} & \lambda_{k+2} & \dots & \lambda_{2k+1} \end{vmatrix},$$

$$[\nu_0, \nu_1, ..., \nu_{2k}], \quad [\nu_1, \nu_2, ..., \nu_{2k+1}] \quad (3.2,10)$$

are non-negative for k = 0, 1, ...n – 1. Thus, we explain &define $\lambda_{2n}, \nu_{2n}, \lambda_{2n+1}, \nu_{2n+1}$. We have undetermined λ_{2n}

$$[\lambda_0, \lambda_1, ..., \lambda_{2n}] = \lambda_{2n}[\lambda_0, \lambda_1, ..., \lambda_{2n-2}] + P, \quad (3.2,11)$$

Here, P is a polynomial in $\lambda_0, \lambda_1, ..., \lambda_{2n-1}$; and similarly for $[\nu_0, \nu_1, ..., \nu_{2n}]$. Since zero is smaller than $[\lambda_0, \lambda_1, ..., \lambda_{2n-2}]$ and $[\nu_0, \nu_1, ..., \nu_{2n-2}]$ so, we can choose λ_{2n} and ν_{2n} non-negative and so large that $\lambda_{2n} - \nu_{2n} = \mu_{2n}$ as-

$$[\lambda_0, \lambda_1, ..., \lambda_{2n}] > 0, \quad [\nu_0, \nu_1, ..., \nu_{2n}] > 0$$

It has been observed that (3.2,11) holds with all subscripts increased by unity, P now being a polynomial in $\lambda_1, \lambda_2, ..., \lambda_{2n}$, a similar equation holding for the ν_k. With λ_{2n} and ν_{2n} now determined we proceed exactly as above to determine $\lambda_{2n+1}$and ν_{2n+1}. Thus, the induction is completed. Following Axiom 3.1(b) if the determinants (3.2,9) and (3.2,10) are non-negative for k = 0, 1, 2, ... equations (3.2, 7) and (3.2, 8) have closely related solutions $\beta(t)$ and $\gamma(t)$ respectively, so that when $\alpha(t)$ is explained & defined as $\beta(t) - \gamma(t)$ our proof is proved.

Stieltjes is mentioned to prove there exists a function, not a constant, all the moments of which are zero. It too follows from Axiom 3.2(a). For, by this result, there is an existence of a non-constant function $\alpha(t)$ such as-

$$\int_0^\infty t^n d\alpha(t) = 1 \qquad (n = 1)$$

$$(n = 0, 2, 3, 4, ...)$$

$$\int_0^\infty |d\alpha(t)| < \infty .$$

Putting;

$$\beta(t) = \alpha(t^{1/2}),$$

we form-

$$\int_0^\infty t^n d\beta(t) = \int_0^\infty t^{2n} d\alpha(t) = 0 \qquad (n = 0, 2, 3, 4, ...).$$

The function $\beta(t)$ is the required example .

Conclusion

In this paper the applications of moment problem to various area deals with the stieltjes moment problem related to the hamburger problem.Asignificant and required condition that there should exist an increasing function $\alpha(t)$ such as-

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...),$$

equations should have an increasing solution $\alpha(t)$ with infinitely many points of non-decrease and equations should have an increasing solution $\alpha(t)$ with a finite number of points of non-decrease .

References

1 **Akhiezer N.I.(1965).** The Classical Moment Problem , Hafner, New York.

2 **Boas, R. P., JR. (1939)**. The Stieltjes moment problem for functions of bounded variation. Bul-letin of the American Mathematical Society, vol. 45, pp. 399-404.

3 **Boas, R. P., JR. (1941)**. Functions with positive derivatives. Duke Mathematical Journal, vol. 8, pp. 163-172.

4 **Boas, R. P., JR., and Widder, D. V. (1939)**. The iterated Stieltjes transform. Transactions of the American Mathe-matical Society, vol. 45, pp. 1-72.

5 **Boas, R. P., JR., and Widder, D. V. (1940a)**. An inversion formula for the Laplace integral. Duke Mathematical Journal, vol. 6, pp. 1-26.

6 **Boas, R. P., JR., and Widder, D. V. (1940b)**. Functions with positive differences. Duke Mathematical Journal, vol. 7, pp. 496 503. .

7 **Bray, H. E. (1919)**. Elementary properties of the Stieltjes integral. Annals of Mathematics, vol. 20. Pp.177-186.

8 **Davies, B. and B. Martin (1979).** Numerical inversion of the Laplace transform: a survey and comparison of methods. *J. Computational Physics*, 33, 1-32.

9 **Evans, G.A. and K.C. Chung (2000).** Laplace transform inversion using optimal contours in the complex plane. *Int. J. Comput. Math.* 73, 531-543.

10 **Feller, W. (1939)**. Completely monotone functions and sequences. Duke Mathematical Journal, vol. 5, pp. 662-663.

11 **Gaver, D.P. (1966).** Oberving stochastic processes and approximate transform inversion. *Operations Research* 14, 444-459.

12 **Gruss, G. (1935)**. BemerkungenzurTheorie der voll-bzw. mehrfachmonotonen funk tionen. MathematischeZeitschrift, vol. 39, pp. 732-741..

13 **Hamburger, H. (1921)**. Ober dieRiemannscheFunktionalgleichung der r-Funktion. Mathe-matischeZeitschrift, vol. 10, pp. 240-254.

14 **Hardy, G. H.(1921)**. Notes on some points in the integral calculus, LIV. Messenger of Mathematics, vol. 50, pp. 165-171.

15 **Hardy, G. H., and Littlewood, J. E. (1928)**. Some properties of fractional integrals. MathematischeZeitschrift, vol. 27, pp. 565-606.

16 **Hildebrandt, T. H. (1938)**. Stieltjes integrals of the Riemann type. American Mathematical Monthly, vol. 45, pp. 265-277.

17 **Hildebrandt, T. H., and Schoenberg, I. J. (1933)**. On linear functional operations and the moment problem for a finite interval in one or several dimensions. Annals of Mathematics, vol. 34, pp. 317-328.

18 **Riesz, M. (1922)**. Sur le probleme des moments. ArkivForMatematik, AstronomiOehFysik, vol. 17, no. 16, pp. 1-52.