IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 7 Issue 9, September 2020
ISSN (Online) 2348 – 7968 | Impact Factor (2020) – 6.72
www.ijiset.com

Boundedness of Two Dimensional Fourier-Laplace Transform and Fourier-Finite Mellin Transform And its Applications

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Abstract

Integral transforms are always used for various purposes. It is related to the various fields of mathematics, physics and engineering. There are large number of integral transforms and they are studied by various authors due to its wide spread applicability in every field of science and technology. So for applying these integral transforms it is very necessary to develop these integral transforms and study their various properties.

In this paper we have developed Two Dimensional Fourier-Laplace Transform and Two Dimensional Fourier-Finite Mellin Transform in the distributional manner and discussed its boundedness property. Apart from this we have discussed an application of Fourier-Laplace Transform.

Keywords: *Fourier Transform, Laplace Transform, Mellin Transform, Generalized function, Testing function space.*

1. Introduction

Integral transforms are widely used in many areas of science, engineering and technology. Integral transform is a mathematical operator that produces a new function $f(y)$ by integrating the product of an existing function $F(x)$ and a so-called kernel function $K(x,y)$ between suitable limits. This process is called the transformation which is symbolized by the equation

$$f(y)=\int K(x,y)F(x)\,dx.$$

There are several integral transforms which are commonly named by the scientists who introduced them. Out of them one important transform is Laplace transform which invented by the French mathematician Pierre-Simon Laplace in 1749-1827. Today it is frequently used by electrical engineers in the field of mechanical electronic circuit problems as well as in the solution of differential equations.

Another important integral transform is a Fourier transform is named in the honor of Joseph Fourier in 1768-1830. Fourier transform is one of the most frequently used tools in signal processing, image processing and many other fields of science and technology [1]. Again we have taken one important and most common integral transform in our study that is Mellin integral transform which introduced by Robert Hjalmar Mellin in 1854-1933. The Mellin transform is a basic tool in Mathematics and Physics. Mellin transform has many applications in quantum calculus [2], agriculture, medical field, statistics and signal processing. It is also used in solution of fractional differential equation [3]. Due to these wide applications of the above transforms we have developed new integral transforms that is Two dimensional Fourier-Laplace transform and Two dimensional Fourier-Finite Mellin Transform and studied them by describing its some properties.

The plan of this paper is as follows:

Some definitions are given in section 2. Section 3 and section 4 gives the definition of Distributional Generalized Fourier-Laplace Transform and Distributional Generalized Fourier-Finite Mellin Transform respectively. Some testing function spaces which are required for development of these transforms are describing in section 5. In section 6 Boundedness theorem for Two dimensional Fourier-Laplace transform is proved and Boundedness theorem for Two dimensional Fourier-Finite Mellin transform is proved in section 7. Finally Conclusions are given in section 8.

The notations and Terminologies are as per Zemanian [4], [5].

2. Definitions

The Two Dimensional Fourier-Laplace transform with parameters $s,\ u,\ p,\ v,$ of $f\left(t,z,x,y\right)$ is defined as,

$$2DFL\left\{f\left(t,z,x,y\right)\right\}=F\left(s,u,p,v\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f\left(t,z,x,y\right)e^{-i\left\{\left(st+uz\right)-i\left(px+vy\right)\right\}}dtdzdxdy \qquad (2.1)$$

where the kernel $K\left(s,u,p,v\right)=e^{-i\left\{\left(st+uz\right)-i\left(px+vy\right)\right\}}$

The Two Dimensional Fourier-Finite Mellin transform with parameters $s,\ u,\ p,\ v$ of $f\left(t,z,x,y\right)$ is defined as,

$$2DFM_{f}\left\{f\left(t,z,x,y\right)\right\}=F\left(s,u,p,v\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{a}\int_{0}^{a}f\left(t,z,x,y\right)K\left(s,u,p,v\right)dtdzdxdy \qquad (2.2)$$

where the kernel $K\left(s,u,p,v\right)=e^{-i\left(st+uz\right)}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}}-y^{v-1}\right)$

3. Distributional Generalized Two Dimensional Fourier-Laplace Transform (2DFLT)

For $f\left(t,z,x,y\right)\in FL^{*}_{a,b,c,d,\alpha}$, where $FL^{*}_{a,b,c,d,\alpha}$ is the dual space of $FL_{a,b,c,d,\alpha}$. It contains all distributions of compact support. The distributional two dimensional Fourier-Laplace transform is a function of $f\left(t,z,x,y\right)$ is defined as,

$$FL\left\{f\left(t,z,x,y\right)\right\}=F\left(s,u,p,v\right)=\left\langle f\left(t,z,x,y\right),\phi\left(t,z,x,y,s,u,p,v\right)\right\rangle, \qquad (3.1)$$

where $\phi\left(t,z,x,y,s,u,p,v\right)=e^{-i\left\{\left(st+uz\right)-i\left(px+vy\right)\right\}}$ and for each fixed $t\left(0<t<\infty\right)$, $z\left(0<z<\infty\right)$, $x\left(0<x<\infty\right)$ and $y\left(0<y<\infty\right)$. Also $s>0,\ u>0,\ \ p>0$ and $v>0$. The right hand side of (3.1) has a sense as an application of $f\left(t,z,x,y\right)\in FL^{*}_{a,b,c,d,\alpha}$ to $\phi\left(t,z,x,y,s,u,p,v\right)\in FL_{a,b,c,d,\alpha}$.

4. Distributional Generalized Two Dimensional Fourier-Finite Mellin Transform

For $f\left(t,z,x,y\right)\in FM^{*}_{f,b,c,d,e,\alpha}$, where $FM^{*}_{f,b,c,d,e,\alpha}$ is the dual space of $FM_{f,b,c,d,e,\alpha}$. It contains all distributions of compact support. The distributional two dimensional Fourier-Finite Mellin transform is a function of $f\left(t,z,x,y\right)$ is defined as,

$$FM_{f}\left\{f\left(t,z,x,y\right)\right\}=F\left(s,u,p,v\right)=\left\langle f\left(t,z,x,y\right),\phi\left(t,z,x,y,s,\ u,p,v\right)\right\rangle, \qquad (4.1)$$

where $\phi\left(t,z,x,y,s,u,p,v\right)=e^{-i\left(st+uz\right)}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}}-y^{v-1}\right)$ and for each fixed $t\left(0<t<\infty\right)$, $z\left(0<z<\infty\right)$, $x\left(0<x<a\right)$ and $y\left(0<y<a\right)$. Also $s>0,\ u>0,\ \ p>0$ and $v>0$ The right hand side of (4.1) has a sense as an application of $f\left(t,z,x,y\right)\in FM^{*}_{f,b,c,d,e,\alpha}$ to $\phi\left(t,z,x,y,s,u,p,v\right)\in FM_{f,b,c,d,e,\alpha}$.

Now for the development of these transforms it is necessary to define some testing function spaces for both the above transforms, so we define these in the following section.

5. Testing function spaces

5.1 The Space $FL_{a,b,c,d,\alpha}$ (S_α -type space)

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, the space $FL_{a,b,c,d,\alpha}$ is given by,

$$FL_{a,b,c,d,\alpha} = \{\phi : \phi \in E_+ / \gamma_{a,b,c,d,k,r,q,m,l,n}\left[\phi(t,z,x,y)\right] = \sup_{I_1}\left|t^k z^r K_{a,b}(x) R_{c,d}(y) D_t^l D_x^q D_z^n D_y^m \left[\phi(t,z,x,y)\right]\right| \le C_{l,q,n,m} A^k k^{k\alpha} B^r r^{r\alpha}\}$$

where the constants A, B and $C_{l,q,n,m}$ depend on the testing function ϕ.

5.2 The Space $FM_{f,b,c,d,e,\alpha}$ (S_α -type space)

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, the space $FM_{f,b,c,d,e,\alpha}$ is given by,

$$FM_{f,b,c,d,e,\alpha} = \{\phi : \phi \in E_+ / \gamma_{b,c,d,e,k,r,q,m,l,n}\left[\phi(t,z,x,y)\right] = \sup_{I_1}\left|t^k z^r \lambda_{b,c}(x) \mu_{d,e}(y) x^{q+1} y^{m+1} D_t^l D_x^q D_z^n D_y^m \left[\phi(t,z,x,y)\right]\right| \le C_{l,q,n,m} A^k k^{k\alpha} B^r r^{r\alpha}\}$$

For each $k, r, q, m, l, n = 0,1,2,..........$

where $\lambda_{b,c}(x) = \begin{cases} x^{+b} & 0 < x < 1 \\ x^{+c} & 1 < x < a \end{cases}$ and $\mu_{d,e}(y) = \begin{cases} y^{+d} & 0 < y < 1 \\ y^{+e} & 1 < y < a \end{cases}$

where the constants A, B and $C_{l,q,n,m}$ depend on the testing function ϕ.

6. Boundedness Theorem for Two Dimensional Fourier-Laplace Transform

6.1. Theorem

Let $f(t,z,x,y) \in FL^*_{a,b,c,d,\alpha}$ and $F(s,u,p,v) = FL\{f(t,z,x,y)\} = \left\langle f(t,z,x,y), e^{-i\{(st+uz)-i(px+vy)\}}\right\rangle$, $a < \operatorname{Re} p < b$, $c < \operatorname{Re} v < d$, $s > 0$, $u > 0$. Let $\operatorname{supp} f(t,z,x,y) \in S_A \cap S_B$, such that $S_A = \{t,z : t,z \in R^n, |t|,|z| \le A, A > 0\}$ and $S_B = \{x,y : x,y \in R^n, |x|,|y| \le B, B > 0\}$, then for each $\varepsilon > 0$, $\delta > 0$, $\xi > 0$ and $\tau > 0$ there exist a constant $C > 0$ and a non-negative integer k such that

$$\left|F(s,u,p,v)\right| \le C\left(1+|s|^l\right)\left(1+|u|^n\right) e^{|s|(A+\varepsilon)} e^{|u|(A+\xi)} \max_{0 \le q \le k} \max_{0 \le m \le k} e^{-p(B+\delta)} e^{-v(B+\tau)}$$

Proof: Suppose that $\operatorname{supp} f(t,z,x,y) \in S_A \cap S_B$ and let $\varepsilon > 0$, $\delta > 0$, $\xi > 0$ and $\tau > 0$. Choose $\rho \in D(R^n)$, $\sigma \in D(R^n)$ such that $\rho(t) = 1$, $\sigma(z) = 1$ on a neighborhood of S_A, $\operatorname{supp}\rho \subset S_{A+\varepsilon}$ and $\operatorname{supp}\sigma \subset S_{A+\xi}$. Since $f \in FL^*_{a,b,c,d,\alpha}$ and in view of the boundedness property of the generalized functions, there exist a constant C and a non-negative integer k such that $\left|F(s,u,p,v)\right| = \left|\left\langle f(t,z,x,y), e^{-i\{(st+uz)-i(px+vy)\}}\right\rangle\right|$

$$= \left|\left\langle f(t,z,x,y), \rho(t)\sigma(z) e^{-i\{(st+uz)-i(px+vy)\}}\right\rangle\right|$$

$$\le C_1 \max_{\substack{0\le l\le k \\ 0\le q\le k}} \max_{\substack{0\le n\le k \\ 0\le m\le k}} \left[\gamma_{a,b,c,d,k,r,q,m,l,n}\left(\rho(t)\sigma(z)e^{-i\{(st+uz)-i(px+vy)\}}\right)\right]$$

$$\le C_1 \max_{\substack{0\le l\le k \\ 0\le q\le k}} \max_{\substack{0\le n\le k \\ 0\le m\le k}} \sup_{I_1}\left|t^k z^r e^{ax} e^{cy} D_t^l D_x^q D_z^n D_y^m \rho(t)\sigma(z) e^{-i\{(st+uz)-i(px+vy)\}}\right|$$

$$\le C_1 \max_{\substack{0\le l\le k \\ 0\le q\le k}} \max_{\substack{0\le n\le k \\ 0\le m\le k}} \sup_{I_1}\left|t^k z^r \sum_h \binom{l}{h} D_t^{l-h} \sum_g \binom{n}{g} D_z^{n-g}\left(\rho(t)\sigma(z)\right) D_t^h D_z^g e^{-i(st+uz)} e^{ax} e^{cy} D_x^q D_y^m e^{-(px+vy)}\right|$$

$$\le C_1 \max_{\substack{0\le l\le k \\ 0\le q\le k}} \max_{\substack{0\le n\le k \\ 0\le m\le k}} \sup_{I_1}\left|t^k z^r \sum_h \binom{l}{h} D_t^{l-h} \sum_g \binom{n}{g} D_z^{n-g}\rho(t)\sigma(z) \cdot (-is)^h (-iu)^g e^{-i(st+uz)} e^{ax+cy} (-p)^q (-v)^m e^{-(px+vy)}\right|$$

$$\le C_1 t^k z^r e^{|s|(A+\varepsilon)} e^{|u|(A+\xi)} \max_{\substack{0\le l\le k \\ 0\le q\le k}} \max_{\substack{0\le n\le k \\ 0\le m\le k}} \left(\sum_h \binom{l}{h}|s|^h\right)\left(\sum_g \binom{n}{g}|u|^g\right)\left|p^q v^m e^{-\{(p-a)x+(v-c)y\}}\right|$$

$$\le C_1\left(1+|s|^l\right)\left(1+|u|^n\right) t^k z^r e^{|s|(A+\varepsilon)} e^{|u|(A+\xi)} \max_{0\le q\le k} \max_{0\le m\le k}\left|p^q v^m e^{-\{(p-a)x+(v-c)y\}}\right|$$

$$\le C_1\left(1+|s|^l\right)\left(1+|u|^n\right) e^{|s|(A+\varepsilon)} e^{|u|(A+\xi)} C' \max_{0\le q\le k} \max_{0\le m\le k} e^{-p(B+\delta)} e^{-v(B+\tau)},$$

where $C' = t^k z^r \left|e^{ax+by} p^q v^m\right|$

$$\le C\left(1+|s|^l\right)\left(1+|u|^n\right) e^{|s|(A+\varepsilon)} e^{|u|(A+\xi)} \max_{0\le q\le k} \max_{0\le m\le k} e^{-p(B+\delta)} e^{-v(B+\tau)}, \qquad \text{where } C = C'C_1$$

7. Boundedness Theorem for Two Dimensional Fourier-Finite Mellin Transform

7.1. Theorem

Let $f(t,z,x,y) \in FM^{*}_{f,b,c,d,e,\alpha}$ and

$$F(s,u,p,v) = FM_f\{f(t,z,x,y)\} = \left\langle f(t,z,x,y), e^{-i(st+uz)}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}} - y^{v-1}\right)\right\rangle,$$

$a < \operatorname{Re} p < b$, $c < \operatorname{Re} v < d$, $s > 0$, $u > 0$. Let $\operatorname{supp} f(t,z,x,y) \in S_A \cap S_B$, such that $S_A = \{t,z : t,z \in R^n, |t|,|z| \le A, A > 0\}$ and $S_B = \{x,y : x,y \in R^n, |x|,|y| \le B, B > 0\}$, then for each $\varepsilon > 0$, $\delta > 0$, $\xi > 0$ and $\tau > 0$ there exist a constant $C > 0$ and a non-negative integer k such that

$$|F(s,u,p,v)| \le C\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\le q\le k} \max_{0\le m\le k}\left[(B+\delta)^{-p} - (B+\delta)^{p}\right](B+\tau)^{v}$$

Proof: Suppose that $\operatorname{supp} f(t,z,x,y) \in S_A \cap S_B$ and let $\varepsilon > 0$, $\delta > 0$, $\xi > 0$ and $\tau > 0$. Choose $\rho \in D(R^n)$, $\sigma \in D(R^n)$ such that $\rho(t) = 1$, $\sigma(z) = 1$ on a neighborhood of S_A, $\operatorname{supp}\rho \subset S_{A+\varepsilon}$ and $\operatorname{supp}\sigma \subset S_{A+\xi}$. Since $f \in FM^{*}_{f,b,c,d,e,\alpha}$ and in view of the boundedness property of the generalized functions, there exist a constant C and a non-negative integer k such that $|F(s,u,p,v)| = \left|\left\langle f(t,z,x,y), e^{-i(st+uz)}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}} - y^{v-1}\right)\right\rangle\right|$

$$= \left|\left\langle f(t,z,x,y), \rho(t)\sigma(z)e^{-i(st+uz)}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}} - y^{v-1}\right)\right\rangle\right|$$

$$\le C_1 \max_{\substack{0\le l\le k\\ 0\le q\le k}} \max_{\substack{0\le n\le k\\ 0\le m\le k}}\left[\gamma_{b,c,d,e,k,r,q,m,l,n}\left(\rho(t)\sigma(z)e^{-i(st+uz)}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}} - y^{v-1}\right)\right)\right]$$

$$\le C_1 \max_{\substack{0\le l\le k\\ 0\le q\le k}} \max_{\substack{0\le n\le k\\ 0\le m\le k}} \sup_{I_1}\left|t^k z^r \lambda_{b,c}(x)\mu_{d,e}(y)x^{q+1}y^{m+1} D_t^l D_x^q D_z^n D_y^m \rho(t)\sigma(z)e^{-i(st+uz)}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}} - y^{v-1}\right)\right|$$

$$\le C_1 \max_{\substack{0\le l\le k\\ 0\le q\le k}} \max_{\substack{0\le n\le k\\ 0\le m\le k}} \sup_{I_1}\left|t^k z^r \sum_h \binom{l}{h} D_t^{l-h} \sum_g \binom{n}{g} D_z^{n-g}\left(\rho(t)\sigma(z)\right) \cdot D_t^h D_z^g e^{-i(st+uz)}\lambda_{b,c}(x)\mu_{d,e}(y)x^{q+1}y^{m+1}D_x^q D_y^m\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left(\frac{a^{2v}}{y^{v+1}} - y^{v-1}\right)\right|$$

$$\leq C_1 \max_{\substack{0\leq l\leq k\\0\leq q\leq k}} \max_{\substack{0\leq n\leq k\\0\leq m\leq k}} \sup_{I_1}\left|t^k z^r \sum_h \binom{l}{h} D_t^{l-h} \sum_g \binom{n}{g} D_z^{n-g} \rho(t)\sigma(z)\right.$$
$$(-is)^h(-iu)^g e^{-i(st+uz)}\lambda_{b,c}(x)\mu_{d,e}(y)x^{q+1}y^{m+1}\left\{a^{2p}P(-p-q)x^{-p-q-1}-P(p-q)x^{p-q-1}\right\}$$
$$\left.\left\{a^{2v}P(-v-m)y^{-v-m-1}-P(v-m)y^{v-m-1}\right\}\right|$$

where $P(-p-q)$ is a polynomial in $(-p-q)$ and $P(-v-m)$ is a polynomial in $(-v-m)$ etc.

$$\leq C_1 t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{\substack{0\leq l\leq k\\0\leq q\leq k}} \max_{\substack{0\leq n\leq k\\0\leq m\leq k}} \left(\sum_h \binom{l}{h}|s|^h\right)\left(\sum_g \binom{n}{g}|u|^g\right).$$
$$\left|\lambda_{b,c}(x)\mu_{d,e}(y)\left\{\left(\frac{a}{x}\right)^{2p}P(-p-q)-P(p-q)\right\}x^p\left\{\left(\frac{a}{y}\right)^{2v}P(-v-m)-P(v-m)\right\}y^v\right|$$

$$\leq C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\leq q\leq k} \max_{0\leq m\leq k}\left|\lambda_{b,c}(x)\mu_{d,e}(y)\right.$$
$$\left.\left\{\left(\frac{a}{x}\right)^{2p}P(-p-q)-P(p-q)\right\}x^p\left\{\left(\frac{a}{y}\right)^{2v}P(-v-m)-P(v-m)\right\}y^v\right|$$

$$\leq C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\leq q\leq k} \max_{0\leq m\leq k}\left|\left\{\lambda_{b,c}(x)a^{2p}P(-p-q)x^{-p}\right.\right.$$
$$\left.\left.-\lambda_{b,c}(x)P(p-q)x^p\right\}\left\{\mu_{d,e}(y)a^{2v}P(-v-m)y^{-v}-\mu_{d,e}(y)P(v-m)y^v\right\}\right|$$

$$\leq C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\leq q\leq k} \max_{0\leq m\leq k}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)\right.\right.$$
$$\left.a^{2(p+v)}P(-p-q)P(-v-m)x^{-p}y^{-v}\right\}-\left\{\lambda_{b,c}(x)\mu_{d,e}(y)a^{2p}P(-p-q)P(v-m)x^{-p}y^v\right\}$$
$$\left.-\left\{\lambda_{b,c}(x)\mu_{d,e}(y)a^{2v}P(p-q)P(-v-m)x^p y^{-v}\right\}+\left\{\lambda_{b,c}(x)\mu_{d,e}(y)P(p-q)P(v-m)x^p y^v\right\}\right|$$

$$\leq C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\leq q\leq k} \max_{0\leq m\leq k}\left|\lambda_{b,c}(x)\mu_{d,e}(y)a^{2p}P(-p-q)x^{-p}\right.$$
$$\left.\left\{a^{2v}P(-v-m)y^{-v}-P(v-m)y^v\right\}-\lambda_{b,c}(x)\mu_{d,e}(y)P(p-q)x^p\left\{a^{2v}P(-v-m)y^{-v}-P(v-m)y^v\right\}\right|$$

$$\leq C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\leq q\leq k} \max_{0\leq m\leq k}\left|\left\{a^{2v}P(-v-m)y^{-v}-P(v-m)y^v\right\}\right.$$
$$\left.\left\{\lambda_{b,c}(x)\mu_{d,e}(y)a^{2p}P(-p-q)x^{-p}-\lambda_{b,c}(x)\mu_{d,e}(y)P(p-q)x^p\right\}\right|$$

$$\leq C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)} \max_{0\leq q\leq k} \max_{0\leq m\leq k}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)a^{2p}P(-p-q)x^{-p}\right\}\right.$$
$$\left.\left\{a^{2v}P(-v-m)y^{-v}-P(v-m)y^v\right\}\right|-C_1\left(1+|s|^l\right)\left(1+|u|^n\right)t^k z^r e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}$$

$$\max_{0\le q\le k}\max_{0\le m\le k}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)P(p-q)x^{p}\right\}\left\{a^{2v}P(-v-m)y^{-v}-P(v-m)y^{v}\right\}\right|$$

$$\le C_1\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}t^{k}z^{r}\max_{0\le q\le k}\max_{0\le m\le k}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)a^{2p}P(-p-q)x^{-p}\right\}\right.$$

$$\left.\left\{\left(\frac{a}{y}\right)^{2v}P(-v-m)-P(v-m)\right\}y^{v}\right|-C_1\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)t^{k}z^{r}e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}$$

$$\max_{0\le q\le k}\max_{0\le m\le k}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)P(p-q)x^{p}\right\}\left\{\left(\frac{a}{y}\right)^{2v}P(-v-m)-P(v-m)\right\}y^{v}\right|$$

$$\le C_1\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}C'\max_{0\le q\le k}\max_{0\le m\le k}(B+\delta)^{-p}(B+\tau)^{v}$$

$$-C_1\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}C''\max_{0\le q\le k}\max_{0\le m\le k}(B+\delta)^{p}(B+\tau)^{v}$$

where $C'=t^{k}z^{r}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)a^{2p}P(-p-q)\right\}\left\{\left(\frac{a}{y}\right)^{2v}P(-v-m)-P(v-m)\right\}\right|$

and $C''=t^{k}z^{r}\left|\left\{\lambda_{b,c}(x)\mu_{d,e}(y)P(p-q)\right\}\left\{\left(\frac{a}{y}\right)^{2v}P(-v-m)-P(v-m)\right\}\right|$

$$\le C\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}\max_{0\le q\le k}\max_{0\le m\le k}(B+\delta)^{-p}(B+\tau)^{v}$$

$$-C\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}\max_{0\le q\le k}\max_{0\le m\le k}(B+\delta)^{p}(B+\tau)^{v}$$

Where $C=C'C_1=C''C_1$

$$\le C\left(1+|s|^{l}\right)\left(1+|u|^{n}\right)e^{|s|(A+\varepsilon)}e^{|u|(A+\xi)}\max_{0\le q\le k}\max_{0\le m\le k}\left[(B+\delta)^{-p}-(B+\delta)^{p}\right](B+\tau)^{v}$$

8. Application of Fourier-Laplace Transform

In this section we solve Wave equation by applying Fourier-Laplace transform.
We know that wave equation is

$$\frac{\partial^2}{\partial t^2}f(t,x)=c^2\frac{\partial^2}{\partial x^2}f(t,x)\quad \text{i.e.}\quad f_{tt}(t,x)=c^2 f_{xx}(t,x) \tag{8.1}$$

Now we know that

$$FL\{f(t,x)\}=F(s,p)=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(st-ipx)}f(t,x)\,dtdx$$

$$FL\{f_x(t,x)\}(s,p)=pFL\{f(t,x)\}-k$$

$$FL\{f_{xx}(t,x)\}(s,p)=p^2FL\{f(t,x)\}-pk$$

$$FL\{f_t(t,x)\}(s,p) = isFL\{f(t,x)\} - k$$
$$FL\{f_{tt}(t,x)\}(s,p) = -s^2 FL\{f(t,x)\} - isk$$

Now taking Fourier-Laplace transform on both sides of equation (8.1) we get,

$$FL\{f_{tt}(t,x)\} = c^2 FL\{f_{xx}(t,x)\}$$
$$\Rightarrow -s^2 FL\{f(t,x)\} - isk = c^2\left\{p^2 FL\{f(t,x)\} - pk\right\}$$
$$\Rightarrow -s^2 FL\{f(t,x)\} - isk = c^2 p^2 FL\{f(t,x)\} - c^2 pk$$
$$\Rightarrow c^2 p^2 FL\{f(t,x)\} + s^2 FL\{f(t,x)\} = c^2 pk - isk$$
$$\Rightarrow \left(c^2 p^2 + s^2\right) FL\{f(t,x)\} = k\left(c^2 p - is\right)$$
$$\therefore FL\{f(t,x)\} = \frac{k\left(c^2 p - is\right)}{\left(c^2 p^2 + s^2\right)}$$

So, $$f(t,x) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty}\int_{c-i\infty}^{c+i\infty} e^{i(st-ipx)} FL\{f(t,x)\}\,dsdp$$

$$\therefore f(t,x) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty}\int_{c-i\infty}^{c+i\infty} e^{i(st-ipx)}\left[\frac{k\left(c^2 p - is\right)}{c^2 p^2 + s^2}\right] dsdp$$

9. Conclusion

In this paper we developed Two Dimensional Fourier-Laplace Transform and Two Dimensional Fourier-Finite Mellin Transform in the Distributional sense. The main aim of this paper was to discuss Boundedness Property of Two dimensional Fourier-Laplace Transform and Two Dimensional Fourier-Finite Mellin Transform. Lastly we apply Fourier-Laplace transform for finding solution of Wave equation.

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