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Gaussian Radial Basis Function for solving Parabolic Partial Differential Equation

Benaki Lairenjam[1] and Yengkhom Satyendra Singh[2]

[1] Department of Mathematics, REVA University,Bangalore, Karnataka , India

[2] Department of Mathematics, REVA University, Bangalore, Karnataka , India

Abstract
Solving parabolic partial differential equation is a basis of science and engineering. Solution to this equation will be obtained by Gaussian Radial basis Function (GRBF) collocation method. In the first step collocation technique will be applied to the equation that will transforms the parabolic partial differential equation into a nonlinear system of equation, next the integral present in the nonlinear system of equation will be solved using Simpson's 1/3 rule. Then the nonlinear system of equation will be solved to obtain $\mathbf{u(x,t)}$ and the parameter value $\mathbf{p(t)}$.
Keywords: *collocation, ill-posed, parabolic partial differential equation, radial basis function, shape parameter*

1. Introduction

Parabolic partial differential equation is a mathematical form of various science and engineering problems, with boundary conditions whose complexity depends on the complexity of the geometry of the domain [1, 2, 4, 6, 11-13]. Finding an accurate solution to this equation will give on to the advancement of science and engineering. Numerous methods are applied, to solve this equation. The main objective of solving this equation is to attain an accurate and stable solution.

In this paper we will consider a parabolic partial differential equation of the form,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)u + \emptyset(x,t), 0 \le x \le 1, 0 < t \le T \qquad (1)$$

The initial and boundary conditions are,

$$u(x,0) = u_0(x), 0 \le x \le 1 \qquad (2)$$

$$u(0,t) = g_0(t), 0 < t \le T \qquad (3)$$

$$u(1,t) = g_1(t), 0 < t \le T \qquad (4)$$

And subject to the additional condition,

$$\int_0^1 k(x)u(x,t)dx = E(t), \ 0 < t \le T \qquad (5)$$

where, $\emptyset(x,t), u_0(x), g_0(t), g_1(t)$ and $E(t)$ are known functions and T is a constant. It is also assumed that $k(x)$ satisfies,

$$\int_0^1 |k(x)|dx \le \rho \qquad (6)$$

Our focus is to find an accurate and stable solution to equation (1)-(5). Many methods such as Variational method [13], Homotopy method [11,15], Chebyshev Tau approximation using Chebyshev operational matrix [15], Radial Basis Function (RBF) [9,13], Finite Difference Technique [2,6], Adomain decomposition method [16] etc. were used for finding solution to this equation. Finite difference technique is an effective method for finding solution to partial differential equation but has restriction on the stability as stated by Deghan in [2]. The main problems of finite difference method arise in the mesh generation for complex geometry of the irregular domain.

Finite difference technique is an effective method for finding solution to partial differential equation but has restriction on the stability [2]. The main problems of finite difference method arise in the mesh generation for complex geometry of the irregular domain.

Due to the complexity in mesh generation in finite difference method, Radial basis function (RBF) which is a meshless method have been used for finding solution to partial differential equation [1,3,7,9,10,13]. Collocation technique of radial basis function is used to find the solution [9,11]. It is found that the meshless method performs better than the finite difference method which conclude that radial basis function has more advantages than the traditional method like, finite difference method, finite element method and the boundary element method.

Due to the promising result obtain using RBF, equation (1)-(5) will be solved by using GRBF. In the first step the equation will be applied collocation technique, that transform into a nonlinear system of equation. Next the integral present in the nonlinear system of equation will be solved using Simpson’s 1/3. Then the nonlinear system of equation will be solved to obtain $u(x,t)$ and the parameter value $p(t)$.

2. Radial Basis Function Approximation

First, Radial Basis Function (RBF) networks are feed-forward neural network. The network architecture consists of three layers an input layer, a hidden layer and an output layer. The output from the input and the hidden layer is nonlinear whereas the output from the hidden and output layer is linear. Weights are associated between the nodes connecting the hidden and output nodes. The input nodes collect the input information to the hidden nodes. Hidden nodes are also called radial centers which represent clusters in the input space. The output of the hidden node is a function of the Euclidean distance between the input and the radial center. The architecture of the network is given bellow.

Here $x_1, x_2, \dots \dots x_n$ represent the input and $c_1, c_2, \dots \dots c_n$ represent the hidden node or the radial center. Input nodes collect all the input information to the hidden node. At the hidden node the Euclidean distance $(\|x_i - c_j\|, i = 1,2 \dots n, j = 1,2, \dots k)$ between the input and the radial center is calculated. The output from the hidden node is the radial basis function $\emptyset_j(x_i) = \emptyset(\|x_i - c_j\|)$. The radial basis function of the hidden layer also called basis function are given in table 1.

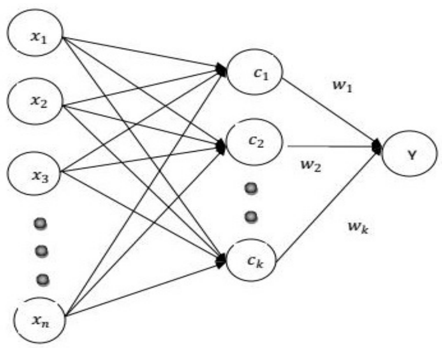

Figure 1. Radial Basis Function Network

Table 1. Radial Basis Function

	Basis Function
Gaussian	$\emptyset(r) = e^{-sr^2}$ where s is the shape parameter and $r = \|x_i - c_j\|$
Multiquadric	$\emptyset(r) = \sqrt{s^2 + r^2}$
Inverse Multiquadric	$\emptyset(r) = 1/\sqrt{s^2 + r^2}$
Thin Plate Splines	$\emptyset(r) = r^2 \log(r)$

Each hidden unit computes the distance between the input vector and its centers. The weights connecting the hidden units to the outputs are used to take linear combinations of the hidden units to produce the final output.

$$Y = f(x_i) = \sum_{j=1}^{k} w_j \emptyset_j(x_i) \qquad (7)$$

For $i = 1 \dots n$, where n is the number of input nodes and $j = 1 \dots k$, where k is the number of hidden nodes. w_j, where $j = 1,2, \dots . k$ is the weight associated between the hidden nodes and the output node. Equation (1) can also be written in matrix form as,

$$A^T W = f \qquad (8)$$

where, $A = \begin{bmatrix} \emptyset_1(x_1) & \emptyset_1(x_2) & \dots & \emptyset_1(x_n) \\ \emptyset_2(x_1) & \emptyset_2(x_2) & \dots & \emptyset_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset_n(x_1) & \emptyset_n(x_2) & \dots & \emptyset_n(x_n) \end{bmatrix}$,

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Equation (8) is a nonlinear system of equation which will be solve by using Gauss Seidel iteration method. The system in equation (8) is ill-posed. The ill posedness can be determined by the condition number, $C(A) = \|A\|\|A^{-1}\|$. Ill posedness is the huge modification in the solution due to a small change in the input data.

3. Statement of the Problem

Consider the partial differential equation (1) – (5) and applying the collocation technique. First, we differentiate equation (5) with respect to t,

$$\int_0^1 k(x)u_t dx = E'(t) \quad (9)$$

Using equation (1) in equation (9), we have

$$\int_0^1 k(x)[u_{xx} + P(t)u + \emptyset(x,t)]dx = E'(t)$$

$$\int_0^1 k(x)\, u_{xx}dx + P(t)\int_0^1 k(x)udx + \int_0^1 k(x)\,\emptyset(x,t)dx = E'(t)$$

$$P(t) = \frac{E'(t) - \int_0^1 k(x)u_{xx}dx - \int_0^1 k(x)\emptyset(x,t)dx}{\int_0^1 k(x)udx} \quad (10)$$

Provided for any $t \in [0,T]$, $\int_0^1 k(x)udx$ exist and is not equal to zero, $\int_0^1 k(x)\,u_{xx}dx$ and $\int_0^1 k(x)\emptyset(x,t)dx$ exists. Now the inverse parabolic partial differential equation (1)-(5) can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + \frac{E'(t) - \int_0^1 k(x)u_{xx}dx - \int_0^1 k(x)\emptyset(x,t)dx}{\int_0^1 k(x)udx} u + \emptyset(x,t), \quad 0 < t < T, 0 < x < 1 \quad (11)$$

$$u(x,0) = u_0(x), 0 \le x \le 1 \quad (12)$$

$$u(0,t) = g_0(t), 0 < t \le T \quad (13)$$

$$u(1,t) = g_1(t), 0 < t \le T \quad (14)$$

Let $G(t) = \int_0^1 k(x)u(x,t)dx$, $R(t) = \int_0^1 k(x)u_{xx}dx$ and $F(t) = \int_0^1 k(x)\emptyset(x,t)dx$.

Now equation (11) reduces to

$$G(t)u_t(x,t) = G(t)u_{xx} + E'(t)u(x,t) - R(t)u(x,t) - F(t)u(x,t) + \emptyset(x,t)G(t) \quad (15)$$

Collocation techniques are used in equation (15), (12), (13) and (14), where $G(t)$, $R(t)$ and $F(t)$ are calculated using Simpson's 1/3 rule. Consider the set of scattered nodes, $C = \{\chi_r = (x_i, t_j), 0 \le x_i \le 1, 0 \le t_j \le T, i, j = 1,2, \dots\dots n, r = 1,2, \dots N = (n+1)^2\}$, where n is the number of sub interval along the x-axis and the t-axis. Let us assume the solution of the problem (15) (12)-(14) as

$$u(\chi) = \sum_{m=1}^{N} \omega_m \emptyset_m(\chi) \quad (16)$$

Where, $\chi = (x,t), \emptyset_m(\chi) = \emptyset(\|\chi - \chi_m\|_2)$, ω_m =weight associated with the input node and the m^{th} hidden node. Applying equation (16) to equation (15), (12), (13) and (14), it reduces to

$$G(t)\sum_{m=1}^{N}\omega_m\frac{\partial\emptyset_m(\chi)}{\partial t}=G(t)\sum_{m=1}^{N}\omega_m\frac{\partial^2\emptyset_m(\chi)}{\partial x^2}+E_t(t)\sum_{m=1}^{N}\omega_m\emptyset_m(\chi)-R(t)\sum_{m=1}^{N}\omega_m\emptyset_m(\chi)-F(t)\sum_{m=1}^{N}\omega_m\emptyset_m(\chi)+\emptyset(\chi)G(t) \quad (17)$$

From the boundary conditions,

$$\sum_{m=1}^{N}\omega_m\emptyset_m(x,0)=u_0(x) \quad (18)$$
$$\sum_{m=1}^{N}\omega_m\emptyset_m(0,t)=g_0(t) \quad (19)$$
$$\sum_{m=1}^{N}\omega_m\emptyset_m(1,t)=g_1(t) \quad (20)$$

We collocate (17) in n(n-1) points, $\chi_k=(x_i,t_j)\in C_4$.

$$G(t)\sum_{m=1}^{N}\omega_m\frac{\partial\emptyset_m(\chi_k)}{\partial t}=G(t)\sum_{m=1}^{N}\omega_m\frac{\partial^2\emptyset_m(\chi_k)}{\partial x^2}+E_t(t)\emptyset_m(\chi_k)-R(t)\sum_{m=1}^{N}\omega_m\emptyset_m(\chi_k)-F(t)\sum_{m=1}^{N}\omega_m\emptyset_m(\chi_k)+\emptyset(\chi_k)G\sum_{m=1}^{N}\omega_m(t) \quad (21)$$

Collocating the initial condition,

$\sum_{m=1}^{N}\omega_m\emptyset_m(x_i,0)=u_0(x_i)$ in n+1 points $\chi_k=(x_i,0)\in C_1, i=1,2,\ldots..n+1$, (22)

Collocating the boundary conditions

$\sum_{m=1}^{N}\omega_m\emptyset_m(0,t_j)=g_0(t_j)$ in n points $\chi_k=(0,t_j)\in C_2, j=1,2,\ldots..n,$ (23)

and,

$\sum_{m=1}^{N}\omega_m\emptyset_m(1,t_j)=g_1(t_j)$ in n points $\chi_k=(1,t_j)\in C_3, i=1,2,\ldots..n.$ (24)

The integrals in equation (17) is obtained by using numerical integral. Here we employed the Simpson's 1/3 rule. Simpson's 1/3 rule at the three nodes x_0, x_1 and x_2 is given by,

$$\int_{x_0}^{x_2}f(x)dx=\frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]-\frac{h^5}{90}f^4(\xi) \quad (25)$$

Equations (21) - (24) will give N((n+1)*(n+1)) nonlinear equations with ω_m unknown which is the transformation of equation (1)-(5). And solving the N equations will determine the values of ω_m which in turn gives us the solution and the value of the parameter $p(t)$ i.e,

$$P(t)=\frac{E'(t)-R(t)-F(t)}{G(t)} \quad (26)$$

Equation (26) is used for finding the unknown parameter.

4. Numerical Example

To evaluate the performance of our proposed model we consider.

Example. Consider the equation (1)-(6), where.

$$f(x)=x+\cos(\pi x)$$
$$g_0(t)=\exp(t)$$
$$g_1(t)=0$$
$$\emptyset(x,t)=\exp(t)[x+\cos(\pi x)+\pi^2\cos(\pi x)]-\exp(t)(1+t^2)[x+\cos(\pi x)]$$
$$E(t)=\exp(t)\left(\frac{3}{4}-\frac{2}{\pi^2}\right)$$
$$K(x)=1+x^2$$

The exact solution is $u(x,t)=\exp(t)(\cos(\pi x)+x)$ and $p(t)=1+t^2$

Table 2. RMSE for U(x,t) and P(t) for N=9

C(shape parameter)	*RMS U(x,t) appox*	*RMS P(t)appox*
0.8	0.0265	0.04026
0.7	0.1786	*0.01794*
0.6	0.0611	0.09026
0.5	0.0318	*0.01794*
0.4	0.0321	*0.01833*
0.3	0.0455	*0.06924*
0.2	0.0355	*0.02334*
0.1	0.0878	*0.19023*
0.09	0.0353	*0.02198*

Table 3. RMSE **for U(x,t) for N=25**

X	*T*	*RMSE U(x,t)* $*10^{-5}$
0.25	0.25	0.0003181
0.5	0.25	0.0240635
0.75	0.25	0.0246408
1	0.25	0.0246408
0.25	0.5	0.0494558
0.5	0.5	0.0496107
0.75	0.5	0.0597844
1	0.5	0.0597844
0.25	0.75	0.0635897
0.5	0.75	0.0725902
0.75	0.75	0.0726125
1	0.75	0.0726125
0.25	1	0.3568954
0.5	1	0.3594263
0.75	1	0.359428
1	1	0.359428

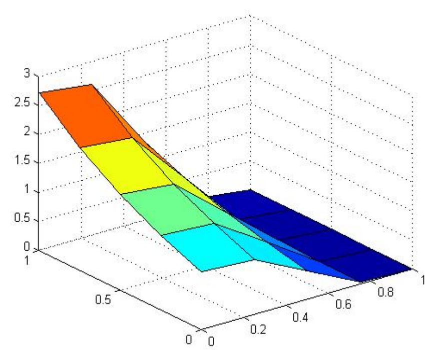

Fig 2. U(x,t) using Radial Basis Function

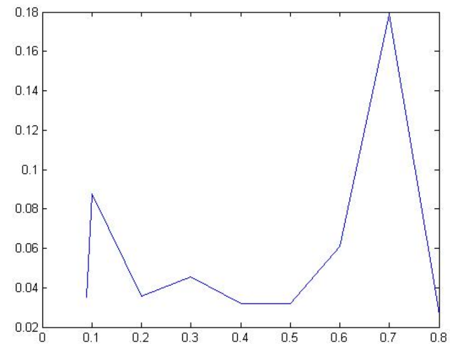

Fig 3. RMSE U(x,t)

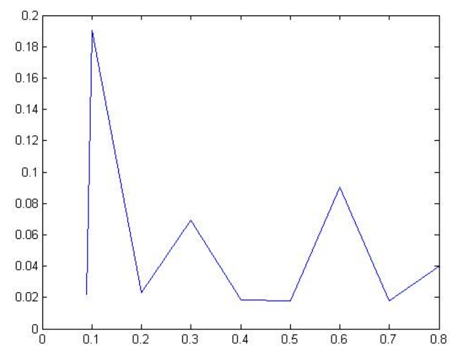

Fig 4. Figure RMSE P(t)

5. Conclusions

The radial basis function is used to solve a parabolic partial differential equation having an unknown parameter. The method is evaluated by taking N=9 and N=25 mesh. From the result we see that the proposed method produces a solution that is much closer to the exact solution for both N=9 and N=25. It is observed that the proposed method correlates well with the analytical solution. Table 2 shows the RMSE of $u(x,t)$ reduces as the shape parameter decreases from 1. In Table 3 we see that the RMSE of $u(x,t)$ decreases as the shape parameter reaches 4. Figure 2 shows the graph of the solution using Radial basis Function. Figure 3 and 4 shows the RMSE of $u(x,t)$ and $p(t)$. In future research we will study the ill posedness of the given problem and to find an optimal shape parameter.

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Dr. Benaki Lairenjam is working as an Assistant Professor in the Department of Mathematics, School of Applied Sciences, Reva University Bangalore India. She completed her PhD from Jamia Millia Islamia (Central University), New Delhi. During her research work she has developed classification models using artificial neural Network, Bayes Theorem, Naïve Bayes classifier and Associative Classifiers for the classification of breast cancer mammography data.