Convergence of Inverse Eigenvalue Problem using the Fibre Bundle Approach with Structure Group $SO(n)$

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1 Abstract

This paper exemplify that an initial n by n singular symmetric matrix of rank two converges to an n by n non-singular symmetric matrix using the direct iterative method. Specifically, a numerical computation is performed with a diagonal matrix which is in the neighborhood of the eigenvalues of the initial three by three singular symmetric matrix of rank two to generate a three dimensional non-singular symmetric matrix. It was observed that the skew-symmetric matrix turns to a zero matrix at the point of convergence.

Keywords – *Convergence, Fibre Bundle, Singular Symmetric Matrix, Skew-Symmetric Matrices, Eigenvalue.*

2 Introduction

Inverse eigenvalue problems are generally pertained with the reconstruction of a physical system from prescribed spectral data. The main aim of an inverse eigenvalue problem thus reduces to the construction of a physical system that maintains a certain specific structure as well as that of the given spectral property[1]. Inverse eigenvalue problems originate in a remarkable variety of applications in both engineering and science. Several authors have used different methods including analytic, algebraic and numerical ones to solve inverse eigenvalue problems especially of various types of non-singular symmetric matrices([2],[3],[4],[5],[6]).
Previous work on inverse eigenvalue problem using the idea of tangent bundles was not practical. The initial matrix for the iteration was not known and therefore the special orthogonal matrix Q could not be ascertained and consequently the skew symmetric matrix K[7]. The work provide information for the initial matrix which in this case is a singular symmetric matrix.
Given the initial singular symmetric matrix, we could obtain the orthogonal and

the skew symmetric matrices for the iteration process. It is expected that the iteration converges to a matrix which is a non-singular symmetric matrix such that its eigenvalues are in the neighborhood of the eigenvalues of the diagonal matrix Λ[8].

3 Methodology

The algorithm described here is more practical and uses information for an initial singular symmetric matrix to generate a non-singular symmetric matrix. The orthogonal and skew symmetric matrices Q and K respectively are used as an initial values for the iteration process. Following from these process, the initial symmetric matrix converges to a non-singular symmetric matrix of the same dimension. Denote the isospectral surface by $M(\Lambda)$. A tangent vector to a point in the fibre $X \in M(\Lambda)$ is of the form $T(X) = XK - KX$. Let $M(\Lambda) = \{Q\Lambda Q^t : Q \in SO(n)\}$, where Λ contains the initial set of distinct eigenvalues. The equation to be solved is given by $X = Q\Lambda Q^t$. A Newton-type method employed is as follows:
$X_{i+1} = Q(A_i)\Lambda Q^t(A_i)$, $i = 1, 2, \ldots,$
where A_i is a singular symmetric matrix, Q, orthogonal matrix whose columns are the normalizes eigenvectors of the matrix A_i and Λ, a diagonal matrix which is in the neighborhood of the eigenvalues of the initial matrix[9]. Linearising iteratively at the tangent space of the lie group which is the lie algebra, we obtain the following equation

$$A_{i+1} = X_{i+1} + X_{i+1}K - KX_{i+1}, i = 1, 2, ...,$$

where K is the skew symmetric matrix which is given by

$$K = \frac{1}{2}(Q - Q^t)$$

At this point, the iteration continues with A_{i+1} being the initial matrix until the iteration converges ($A_{i+1} = X_{i+1}$) at where the eigenvalues of A_{i+1} are in the same neighborhood of the diagonal matrix Λ[10].

4 Preliminaries

In order to present the main results of this research in a sententious way, it is useful to give some preliminary results by [11], which play a fundamental role throughout the rest of the work.

Theorem 4.1 The exponential map

$$exp : so(n) \to SO(n)$$

is well-defined.

Proof.[12] First, we need to prove that if A is skew symmetric matrix, then e^A is a rotation matrix. For this, first check that

$$(e^A)^T = e^{A^T}.$$

Then, since $A^T = -A$, we get

$$(e^A)^T = e^{A^T} = e^{-A},$$

and so

$$(e^A)^T e^A = e^{-A} e^A = e^{-A+A} = e^{0_n} = I_n,$$

and similarly,

$$e^A (e^A)^T = I_n,$$

showing that e^A is orthogonal. Also,

$$det(e^A) = e^{tr(A)},$$

and since A is real skew symmetric, $tr(A) = 0$, and so $det(e^A) = +1$.

Lemma 1 The exponential map

$$exp : so(3) \to SO(3)$$

is given by

$$e^A = \cos\theta I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2$$

if $\theta \neq 0$, with $e^{\theta_3} = I_3$.

Proof.[12] First, prove that

$$A^2 = -\theta^2 + B,$$
$$AB = BA = 0.$$

From the above, deduce that

$$A^3 = -\theta^2 A,$$

and for any $k \geq 0$,

$$A^{4k+1} = \theta^{4k} A,$$
$$A^{4k+2} = \theta^{4k} A^2,$$
$$A^{4k+3} = -\theta^{4k+2} A,$$
$$A^{4k+4} = -\theta^{4k+2} A^2.$$

Finally, we prove the desired result by writing the powers for e^A and regrouping terms so that the power series for *cosine* and *sine* show up.

We state the next lemma without proof. The lemma shows that every symmetric matrix A is of the form e^B.

Lemma 2 For every symmetric matrix B, the matrix e^B is symmetric positive definite. For every symmetric positive definite matrix A, there is a unique symmetric matrix B such that $A = e^B$.

Lemma 3 A nonsingular symmetric matrix can be generated using a singular symmetric matrix as initial matrix in the following algorithm:

$$X_i = Q_i(A_i)\Lambda Q_i^t(A_i) \qquad i = 1, 2, \cdots$$

and

$$A_i = X_i + X_iK_i - K_iX_i$$

Proposition 4.1 If a square matrix A has one row (column) as a scalar multiple of another row (column), then A is a singular matrix and det $A = 0$.

Next we look at the numerical computation of the construction of a 3×3 non-singular symmetric matrix.

Given that $A_{(3,1)} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$, whose eigenvalues are $\lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 0$.

Let $Q_1 = \begin{bmatrix} -0.5774 & 0.7071 & -0.4082 \\ 0.5774 & 0.7071 & 0.4082 \\ -0.5774 & 0 & 0.8165 \end{bmatrix}$, let $\Lambda = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$X_1 = Q_1\Lambda Q_1^t = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$K_1 = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & 0.0649 & 0.0846 \\ -0.0649 & 0 & 0.2041 \\ -0.0846 & -0.2041 & 0 \end{bmatrix}$

$A_1 = X_1 + X_1K_1 - K_1X_1 = \begin{bmatrix} 1.0904 & -2.1196 & 0.5721 \\ -2.1196 & 1.1487 & -1.3083 \\ 0.5721 & -1.3083 & 1.7609 \end{bmatrix}$ $Q_2 = \begin{bmatrix} 0.6519 & -0.5064 & 0.5645 \\ 0.7306 & 0.2198 & -0.6465 \\ 0.2033 & 0.8338 & 0.5132 \end{bmatrix}$

$X_2 = Q_2\Lambda Q_2^t = \begin{bmatrix} 1.7620 & 1.6514 & 1.2420 \\ 1.6514 & 2.5045 & 0.0790 \\ 1.2420 & 0.0790 & -0.2666 \end{bmatrix}$

$K_2 = \frac{1}{2}(Q_2 - Q_2^t) = \begin{bmatrix} 0 & -0.6185 & 0.1806 \\ 0.6185 & 0 & -0.7402 \\ -0.1806 & 0.7482 & 0 \end{bmatrix}$

$A_2 = X_2 + X_2K_2 - K_2X_2 = \begin{bmatrix} 3.3560 & 3.0156 & 0.4349 \\ 3.0156 & 0.5788 & -2.4420 \\ 0.4349 & -2.4420 & 0.0652 \end{bmatrix}$ $Q_3 = \begin{bmatrix} 0.3721 & -0.7870 & 0.4921 \\ -0.7076 & -0.5836 & -0.3984 \\ -0.6007 & 0.1999 & 0.7740 \end{bmatrix}$

$X_3 = Q_3\Lambda Q_3^t = \begin{bmatrix} 0.1766 & -1.7085 & -0.3559 \\ -1.7085 & 1.8207 & 1.5086 \\ -0.3559 & 1.5086 & 2.0027 \end{bmatrix}$

$K_3 = \frac{1}{2}(Q_3 - Q_3^t) = \begin{bmatrix} 0 & -0.0397 & 0.5464 \\ 0.0397 & 0 & -0.2992 \\ -0.5464 & 0.2992 & 0 \end{bmatrix}$

$$A_3 = X_3 + X_3K_3 - K_3X_3 = \begin{bmatrix} 0.4298 & -2.5740 & -0.7827 \\ -2.5740 & 2.8591 & 0.6436 \\ -0.7827 & 0.6436 & 0.7112 \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} 0.5310 & -0.8471 & -0.0227 \\ -0.8143 & -0.5027 & -0.2900 \\ -0.2343 & -0.1724 & 0.9568 \end{bmatrix}$$

$$X_4 = Q_4\Lambda Q_4^t = \begin{bmatrix} 0.4108 & -2.1489 & -0.6653 \\ -2.1489 & 2.4840 & 0.3989 \\ -0.6653 & 0.3989 & 1.1052 \end{bmatrix}$$

$$K_4 = \frac{1}{2}(Q_4 - Q_4^t) = \begin{bmatrix} 0 & -0.0164 & 0.1058 \\ 0.0164 & 0 & -0.0588 \\ -0.1058 & 0.0588 & 0 \end{bmatrix}$$

$$A_4 = X_4 + X_4K_4 - K_4X_4 = \begin{bmatrix} 0.4813 & -2.1963 & -0.6060 \\ -2.1963 & 2.6012 & 0.1014 \\ -0.6060 & 0.1014 & 0.9175 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} -0.8332 & -0.1305 & 0.5374 \\ -0.5003 & -0.2363 & -0.8330 \\ -0.2357 & 0.9629 & -0.1316 \end{bmatrix}$$

$$X_5 = Q_5\Lambda Q_5^t = \begin{bmatrix} 3.0484 & 1.1887 & 0.8406 \\ 1.1887 & 1.6391 & 0.8089 \\ 0.8406 & 0.8089 & -0.6875 \end{bmatrix},$$

$$K_5 = \frac{1}{2}(Q_5 - Q_5^t) = \begin{bmatrix} 0 & 0.1849 & 0.3866 \\ -0.1849 & 0 & -0.8979 \\ -0.3866 & 0.8979 & 0 \end{bmatrix}$$

$$A_5 = X_5 + X_5K_5 - K_5X_5 = \begin{bmatrix} 1.9590 & 1.8914 & 1.0679 \\ 1.8914 & 3.5312 & -0.6653 \\ 1.0679 & -0.6653 & -1.4902 \end{bmatrix}$$

$$Q_6 = \begin{bmatrix} -0.5568 & -0.7544 & -0.3478 \\ -0.8306 & 0.5090 & 0.2258 \\ -0.0067 & -0.4146 & 0.9100 \end{bmatrix}$$

$$X_6 = Q_6\Lambda Q_6^t = \begin{bmatrix} 0.7918 & 2.1553 & -0.6143 \\ 2.1553 & 2.5518 & 0.4386 \\ -0.6143 & 0.4386 & 0.6564 \end{bmatrix}$$

$$K_6 = \frac{1}{2}(Q_6 - Q_6^t) = \begin{bmatrix} 0 & 0.0381 & -0.1705 \\ -0.0381 & 0 & 0.3202 \\ 0.1705 & -0.3202 & 0 \end{bmatrix}$$

$$A_6 = X_6 + X_6K_6 - K_6X_6 = \begin{bmatrix} 0.4179 & 2.3597 & 0.0359 \\ 2.3597 & 2.4353 & 0.6545 \\ 0.0359 & 0.6545 & 1.1468 \end{bmatrix} \quad Q_7 = \begin{bmatrix} -0.5309 & -0.8171 & -0.2246 \\ -0.8260 & 0.5582 & -0.0784 \\ -0.1894 & -0.1439 & 0.9713 \end{bmatrix}$$

$$X_7 = Q_7\Lambda Q_7^t = \begin{bmatrix} 0.5101 & 2.2278 & 0.0664 \\ 2.2278 & 2.4237 & 0.6300 \\ 0.0664 & 0.6300 & 1.0662 \end{bmatrix}$$

$$K_7 = \frac{1}{2}(Q_7 - Q_7^t) = \begin{bmatrix} 0 & 0.004 & -0.0176 \\ -0.0044 & 0 & 0.0328 \\ 0.0176 & -0.038 & 0 \end{bmatrix}$$

$A_7 = X_7 + X_7K_7 - K_7X_7 = \begin{bmatrix} 0.4927 & 2.2282 & 0.1465 \\ 2.2282 & 2.4021 & 0.6356 \\ 0.1465 & 0.6356 & 1.1051 \end{bmatrix}$ $Q_8 = \begin{bmatrix} -0.5306 & -0.8218 & -0.2073 \\ -0.8219 & 0.5587 & -0.1112 \\ -0.2072 & -0.1114 & 0.9719 \end{bmatrix}$

$X_8 = Q_8 \Lambda Q_8^t = \begin{bmatrix} 0.4939 & 2.2267 & 0.1468 \\ 2.2267 & 2.4021 & 0.6354 \\ 0.1468 & 0.6354 & 1.1040 \end{bmatrix}$

$K_8 = \frac{1}{2}(Q_8 - Q_8^t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $A_8 = X_8$

eigenvalues of X_8 are $\lambda_1 = 4, \lambda_2 = -1, \lambda_3 = 1$.
This implies that 3×3 singular symmetric matrix of rank one has generated a three dimensional non-singular symmetric matrix.

5 Main Results

This section seeks a numerical computation to illustrate that a given 3×3 singular symmetric matrix of rank two converges to a non-singular symmetric matrix of the same dimension and that at the point of convergence, the skew symmetric matrix is zero.

Consider $A_{(3,2)} = \begin{bmatrix} 0.1176 & 0.4706 & 5 \\ 0.4706 & 1.8824 & 20 \\ 5 & 20 & 3 \end{bmatrix}$,

whose eigenvalues are $\lambda_1 = -18.1216, \lambda_2 = 0, \lambda_3 = 23.1216$.

let $\Lambda = \begin{bmatrix} -18.09 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 23.09 \end{bmatrix}$

then $Q_1 = \begin{bmatrix} 0.1736 & 0.9701 & 0.1694 \\ 0.6943 & -0.2425 & 0.6776 \\ -0.6985 & -0.0000 & 0.7156 \end{bmatrix}$

$X_1 = Q_1 \Lambda Q_1^t = \begin{bmatrix} 0.1271 & 0.4684 & 49923 \\ 0.4684 & 1.8837 & 19.9694 \\ 4.9923 & 19.9694 & 2.9992 \end{bmatrix}$

$K_1 = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & 0.1379 & 0.4339 \\ -0.1379 & 0 & 0.3388 \\ -0.4339 & -0.3388 & 0 \end{bmatrix}$

$A_1 = X_1 + X_1K_1 - K_1X_1 = \begin{bmatrix} -4.3349 & -10.1309 & 1.1501 \\ -10.1309 & -11.5189 & 20.4833 \\ 1.1501 & 20.4833 & 20.8638 \end{bmatrix}$

$Q_2 = \begin{bmatrix} -0.4122 & -0.9056 & -0.0998 \\ -0.8304 & 0.3284 & 0.4500 \\ 0.3748 & -0.2684 & 0.8874 \end{bmatrix}$

$X_2 = Q_2 \Lambda Q_2^t = \begin{bmatrix} -2.8355 & -7.2323 & 0.7520 \\ -7.2323 & -7.7987 & 14.8501 \\ 0.7520 & 14.8501 & 15.6442 \end{bmatrix}$

$$K_2 = \frac{1}{2}(Q_2 - Q_2^t) = \begin{bmatrix} 0 & -0.0376 & -0.2373 \\ 0.0376 & 0 & 0.3592 \\ 0.2373 & -0.3592 & 0 \end{bmatrix}$$

$$A_2 = X_2 + X_2K_2 - K_2X_2 = \begin{bmatrix} -3.0222 & -4.1654 & 3.0972 \\ -4.1654 & -17.9231 & 8.1175 \\ 3.0972 & 8.1175 & 25.9553 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} -0.2569 & -0.9634 & 0.0765 \\ -0.9491 & 0.2664 & 0.1683 \\ 0.1825 & 0.0294 & 0.9828 \end{bmatrix}$$

$$X_3 = Q_3\Lambda Q_3^t = \begin{bmatrix} -1.0492 & -4.1154 & 2.5847 \\ -4.1154 & -15.6390 & 6.9525 \\ 2.5847 & 6.9525 & 21.6981 \end{bmatrix}$$

$$K_3 = \frac{1}{2}(Q_3 - Q_3^t) = \begin{bmatrix} 0 & -0.0072 & -0.0530 \\ 0.0072 & 0 & 0.0694 \\ 0.0530 & -0.0694 & 0 \end{bmatrix}$$

$$A_3 = X_3 + X_3K_3 - K_3X_3 = \begin{bmatrix} -0.8343 & -4.0312 & 3.5543 \\ -4.0312 & -16.5454 & 4.5593 \\ 3.5543 & 4.5593 & 22.3898 \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} -0.2492 & -0.9599 & 0.1286 \\ -0.9597 & 0.2625 & 0.0999 \\ 0.1296 & 0.0985 & 0.9867 \end{bmatrix}$$

$$X_4 = Q_4\Lambda Q_4^t = \begin{bmatrix} -0.7327 & -4.0327 & 3.5120 \\ -4.0327 & -16.4318 & 4.5253 \\ 3.5120 & 4.5253 & 22.1744 \end{bmatrix}$$

$$K_4 = \frac{1}{2}(Q_4 - Q_4^t) = e^{-0.3} \begin{bmatrix} 0 & -0.0706 & -0.5253 \\ 0.0706 & 0 & 0.6837 \\ 0.5253 & -0.6837 & 0 \end{bmatrix}$$

$$A_4 = X_4 + X_4K_4 - K_4X_4 = \begin{bmatrix} -0.7295 & -4.0339 & 3.5216 \\ -4.0339 & -16.4374 & 4.5008 \\ 3.5216 & 4.5008 & 22.1769 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} 0.2492 & 0.9598 & 0.1291 \\ 0.9598 & -0.2625 & 0.0992 \\ -0.1291 & -0.0992 & 0.9867 \end{bmatrix}$$

$$X_5 = Q_5\Lambda Q_5^t = \begin{bmatrix} -0.7295 & -4.0339 & 3.5216 \\ -4.0339 & -16.4374 & 4.5008 \\ 3.5216 & 4.5008 & 22.1769 \end{bmatrix}$$

$$K_5 = \frac{1}{2}(Q_5 - Q_5^t) = \begin{bmatrix} 0 & 0 & 0.1291 \\ 0 & 0 & 0.0992 \\ -0.1291 & -0.0992 & 0 \end{bmatrix}$$

$$A_5 = X_5 + X_5K_5 - K_5X_5 = \begin{bmatrix} -1.6386 & -4.9641 & 0.1648 \\ -4.9641 & -17.3301 & 0.1506 \\ 0.1648 & 0.1506 & 23.9787 \end{bmatrix} \quad Q_6 = \begin{bmatrix} -0.2784 & -0.9605 & 0.0059 \\ -0.9605 & 0.2784 & 0.0029 \\ 0.0045 & 0.0048 & 1.0000 \end{bmatrix}$$

$$X_6 = Q_6\Lambda Q_6^t = \begin{bmatrix} -1.3917 & -4.8387 & 0.1578 \\ -4.8387 & -16.6870 & 0.1453 \\ 0.1578 & 0.1453 & 230886 \end{bmatrix}$$

$$K_6 = \frac{1}{2}(Q_6 - Q_6^t) = e^{-0.3}\begin{bmatrix} 0 & 0.0038 & 0.7037 \\ -0.0038 & 0 & -0.9366 \\ -0.7037 & 0.9366 & 0 \end{bmatrix}$$

$$A_6 = X_6 + X_6K_6 - K_6X_6 = \begin{bmatrix} -1.3919 & -4.8386 & 0.1451 \\ -4.8386 & -16.6867 & 0.1792 \\ 0.1451 & 0.1792 & 23.0886 \end{bmatrix}$$

$$Q_7 = \begin{bmatrix} -0.2784 & -0.9605 & 0.0052 \\ -0.9605 & 0.2784 & 0.0039 \\ 0.0052 & 0.0039 & 1.0000 \end{bmatrix}$$

$$X_7 = Q_7\Lambda Q_7^t = \begin{bmatrix} -1.3918 & -4.8386 & 0.1451 \\ -4.8386 & -16.6867 & 0.1792 \\ 0.1451 & 0.1792 & 23.0886 \end{bmatrix}$$

$$K_7 = \frac{1}{2}(Q_7 - Q_7^t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Thus } A_7 = X_7$$

eigenvalues of X_7 are $\lambda_1 = -18.09, \lambda_2 = 0.01, \lambda_3 = 23.09$.

6 Conclusion

In the paper, a direct iteration method is utilized to deriving sequences of non-singular symmetric matrices from a given singular symmetric matrix. The research was limited to a three by three singular symmetric matrix of rank two. The paper was started by deriving a special orthogonal matrix Q, and a skew-symmetric matrix K from a given three dimensional singular symmetric matrix of rank two. Q and K were used as initial guess matrices for the iterative process. An assumed diagonal matrix Λ in the neighborhood of the eigenvalues of the given three by three singular symmetric matrix was considered. The outcome of these iterations are sequences of non-singular symmetric matrix X_i for $i = 1, 2, .., 7$. It was established that at the point of convergence, the skew-symmetric matrix is zero and the eigenvalues of X_i had the same eigenvalues as the diagonal entries of Λ.

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