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Slow-fast dynamics of the Hindmarsh-Rose model with threshold policy control under slow-varying periodic exactions

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Abstract
This article focuses on the fast-slow dynamics of the Filippov system with two time scales. A 3D Hindmarsh-Rose model with discontinuous righthand sides under slow-varying periodic external exaction and parametric exaction is proposed here. The exaction term could be view as the slow-varying parameter since the exciting frequency is relatively smaller than the natural frequency. By using slow-fast methods, the system can be divided into fast and slow subsystem. Through making bifurcation analysis on the fast subsystem we can obtain different types of both smooth and nonsmooth bifurcations when varying the value of threshold , such as Hopf bifurcation and nonsmooth homoclinic bifurcation. Further more, the sliding phenomenon can be observed on both quiescent state and spiking state when the nonsmooth bifurcation occurs.
Keywords: *Filippov systems, nonsmooth bifurcations, fast-slow dynamics* .

1. Introduction

Recently, threshold policy control possesses broad research prospects[1-3]. One of the application is the Filippov-type forest-pest model in the ecosystem [4].This policy establish a non-smooth Filippov system for numerical simulation to predict the population of the pests and find the key factors which affect the population through dynamic analysis in order to control the population of the pest. Since the achievement of the policy in pest control, many researcher focuses on it. Zhang[5] proposed a Filippov system of predator-prey model with threshold policy control and analysis the dynamics of it. Based on the policy, Yang[6] proposed a Hindmarsh-Rose model with discontinuous righthand sides and analysis the dynamics of it . Since these model all have nonsmooth vectors field, the more complicated dynamics in such systems remains a challenge for further study.

Up to now, the dynamical system we usually considered mainly owns single-time scale factor while the mutil-time scale coupled factors widely exist in the theoretical research and engineering applications. Fast-slow dynamical behaviors, such as mixed-oscillations[7] and bursting oscillations [8]may appear in such dynamical systems. The traditional nonlinear dynamics can not deal with these system since the different time scales coexisting in them. In order to study the mechanism of the fast-slow dynamics, Izhikevich[9] introduced the fast-slow analysis method which was first put forward by Rinzel[10]. The main idea of this method can be illustrated as follows. Firstly, dividing the system into two subsystem named slow subsystem and fast subsystem respectively. Then analyzing the equilibria states and their bifurcations of the fast subsystems and treat the slow subsystem as an adjustment to the fast subsystems in any period of the whole system. Based on this method, the mechanism of the dynamical behavior can be revealed[11].

Since the above discussion, we take the typical Hindmarsh-Rose model[12] as the research object. By introducing the threshold policy control and slow-varying periodic excitations we establish a Filippov-type system with slow fast coupled system in order to investigate the dynamical behavior under different threshold values. The reminder of this article is organized as follows. In section 2, a mutil-time scales coupled Filippov system is proposed. In section 3, the equilibrium branches and their stability analysis are conducted here . Moreover, both smooth and nonsmooth bifurcation together with their necessary conditions are given here. In section 4, the numeric results of the bursting oscillations in the fast-slow couple system are presented, and the mechanism of the fast-slow dynamics are analyzed. The conclusions of our work are shown in the last section.

2. Mathematic Model

This article takes 3D Filippov-type Hindmarsh-Rose model as an example. By introducing the external exaction and parametric exaction, an nonautomuous system is established. Here are the mathematical expression:

$$\begin{aligned}\dot{x} &= y - ax^3 + bx^2 - fz + sgn(x - MT)I_0 + w \\ \dot{y} &= c - dx^2 - y \\ \dot{z} &= r[s(x - x_\gamma) - z + wx]\end{aligned} \tag{1}$$

where $w = A\sin(\Omega t)$, A and Ω represent the amplitude and frequency of the periodic exactions respectively .When there keeps an order gasp between the natural frequency and the exciting frequency, the methods of divided the fast- slow subsystem and bifurcation analysis can be introduced to investigate the dynamics of the general autonomous system. To eliminate the influence of the slow-varying state variable in the original system,setting the value of parameter $r = 1$ and the frequency of the $\Omega = 0.01$.

3. Bifurcation analysis

Since there exists an order gasp between the frequency of periodic excitations and natural frequency, we can treat the whole excitation as a slow-varying parameter during any period of the natural frequency. By using slow-fast analysis we can tell the fast and the slow subsystem and view the fast subsystem as the general autonomous system. The mechanism of the whole couple system can be obtain by making bifurcation analysis of the fast subsystem. The detailed analysis focus on sliding region and smooth region will be shown as follows.

3.1 Analysis of smooth vector field

According to the equation (1), a nonsmooth surface denoted by $\Sigma = \{(x, y, z) \mid x = MT\}$ could be derived in the dynamical system which will cause the vector field to de divided into two subsystem denoted by $D_+ = \{(x, y, z) \mid x > MT\}$ and $D_- = \{(x, y, z) \mid x < MT\}$ respectively. The responding expression are as follows:

When satisfy $x < MT$:

$$\begin{aligned}\dot{x} &= y - ax^3 + bx^2 + I_0 + w - fz, \\ \dot{y} &= c - dx^2 - y, \\ \dot{z} &= r[s(x - x_\gamma) - z + wx],\end{aligned} \tag{2}$$

When satisfy $x > MT$:

$$\begin{aligned}\dot{x} &= y - ax^3 + bx^2 - I_0 + w - fz, \\ \dot{y} &= c - dx^2 - y, \\ \dot{z} &= r[s(x - x_\gamma) - z + wx],\end{aligned} \tag{3}$$

The equilibrium point of the two subsystem can be expressed uniformly by $E_0 = (X_0, c - dX_0{}^2, wX_0 + sX_0 - sX_\gamma)$, while X_0 satisfy:

$$c - aX_0{}^3 + (b - d)X_0{}^2 - f(s + w)X_0 + fsx_\gamma + sgn(X_0 - MT)I_0 + w = 0 \tag{4}$$

The characteristic equation is as following:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{5}$$

where $a_1 = 3ax^2 - 2bx + r + 1,\ a_2 = 3arx^2 + 3ax^2 - 2brx + frs + frw - 2bx + 2dx + r$
$a_3 = 3arx^2 - 2brx + 2drx + frs + frw$.

according to Routh-Hurwitz criterion, when satisfy $a_1 > 0, a_3 > 0, a_1 a_2 > a_3$, the equilibrium point E_0 is stable and fold bifurcation may occur at $a_3 = 0$ while Hopf bifurcation occur at $a_1 a_2 > a_3$.
Since the system has nonsmooth factors, the vector field is discontinuous at the nonsmooth surface. Therefore, when the trajectory touches the surface, the nonsmooth behaviors occurs. We will analyze the dynamics on the surface in the following.

3.2 Analysis of non-smooth vector field

According to differential inclusion theory, introducing the auxiliary parameter $q \in [0,1]$ to investigate the dynamics of the nonsmooth boundary denoted by $\sum = \{(x, y) \mid x = MT\}$ .The vector field can rewrite as follow:

$$\dot{X}(t) \in \begin{cases} F_+(X), \\ qF_+(X) + (1-q)F_-(X), \\ F_-(X), \end{cases} \tag{6}$$

When $q = 1$, the dynamics of the trajectory in the smooth subregion D_+ is governed by $F_+(X) = S_+$ .When $q = 0$,, the dynamics of the trajectory in the smooth subregion D_- is governed by $F_-(X) = S_-$.
Since the nonsmooth factor only exists in the vector field $\dot{x}$ and only have infect on x, we can use the formulation $\dot{x} = y(t) - ax(t)^3 + bx(t)^2 + w - fz(t) + (2q-1)I_0$ to descript the dynamical behavior on the surface. The trajectory will keep on the surface when $\dot{x} = 0$ while move away from it when $\dot{x} \neq 0$. In order to find the interact between the nonsmooth factor and scale effect, we give the expression about the boundary on the surface since the amplitude is explicit on time and in one direction. The definition is written as follow:

$$\hat{\Sigma} := \bigcup_{q \in [0,1]} \{(t, \psi(t)) \mid \psi(t) = y(t) - fz(t) = aMT^3 - bMT^2 + (1-2q)\delta - w, t \in R\} \tag{7}$$

We can further obtain that:

$$\begin{aligned} \sup \hat{\Sigma} &= \psi(t) = y(t) - fz(t) = aMT^3 - bMT^2 - \delta - w(t), q = 1 \\ \inf \hat{\Sigma} &= \psi(t) = y(t) - fz(t) = aMT^3 - bMT^2 + \delta - w(t), q = 0 \end{aligned} \tag{8}$$

The nonsmooth bifurcations may happen at the boundary of the nonsmooth surface such as the boundary equilibrium bifurcation and gazing-sliding bifurcation of the limit cycle .This article set the parameter as follows: $a = 1, b = 3, c = 2.5, d = 5,\ I = 5, f = 1,\ r = 1, s = -4, x_\gamma = -1.6$. Fixing the amplitude of A to 7.0 to study the dynamics of the system under different threshold value. The equilibrium branch as well as the corresponding bifurcation are shown as follows, see fig.1

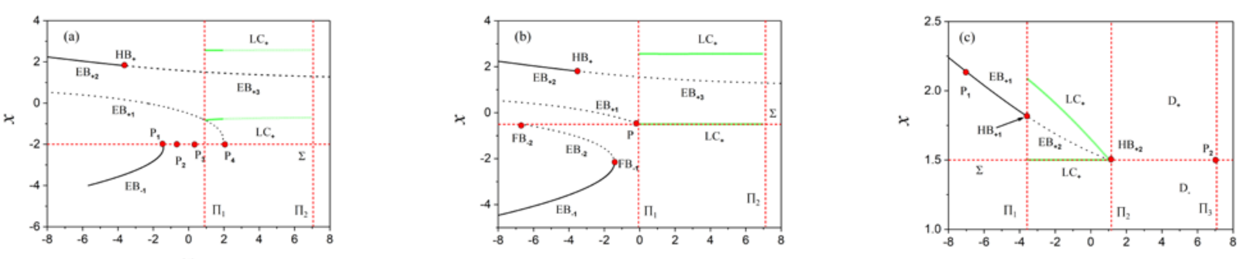

Fig. 1 Equilibrium branches and the corresponding bifurcation diagram for different threshold controls: (a) $MT = -2$ (b) $MT = -0.5$ (c) $MT = 1.5$.

In the above pictures, the black full line is used to represent the stable equilibrium branches while the black dotted line is used to indicated unstable branches .Due to the existence of the nonsmooth surface , the vector field is divided into two smooth region denoted by D_- and D_+ with their boundary which governed by the corresponding dynamics respectively.

When seting $MT=-2$, there keeps a piece of stable equilibrium branch EB_{-1} in the subregion D_- while the equilibrium branch in the subregion D_+ can be divided into three parts. The unstable branch EB_{+1} may disappear at point P_4 $(-1.47,-2)$ .The stable branch EB_{+2} may bifurcate to unstable branch EB_{+3} at subcritical Hopf bifurcation point HB_+ where the parameter w increases to $w=-3.64$. When $w=0.9$ a homoclinic bifurcation occurs at the cross-section Π_1 , which leading to the appearance of the stable limit cycle LC_+ .

When threshold value increase to $MT=-0.5$, the equilibrium branches are divided into two parts by fold bifurcation point FB_{-1} at $(-1.43,-2.1)$ and nonsmooth fold bifurcation point FB_{-2} at $(w=-6.7)$,and named stable equilibrium branch EB_{-1} and unstable equilibrium branch EB_{-2} respectively in the subregion D_- . In its counterpart D_+ ,the equilibrium branch has three parts denoted by unstable equilibrium branch EB_{+1} and stable equilibrium branch EB_{+2} and stable equilibrium branch EB_{+3} which bifurcates from subcriticle Hopf bifurcation point HB_+ at $(-3.5,1.8)$ .Meantime, a stable limit cycle LC_+ caused by the nonsmooth homoclinic bifurcation at point p $(w=-0.1)$ which is evolved from the pseudo equilibrium on the nonsmooth surface in the subregion D_+ .

In Fig.1(c),when threshold increases to $MT=-0.5$, we only focus on the dynamics in the subregion D_+ and the nonsmooth surface .The equilibrium branches in the subregion D_+ are divided into two parts denoted by the unstable equilibrium branch EB_{+2} and the stable counterpart EB_{+1} by the subcriticle Hopf bifurcation HB_+ $(-3.5,1.8)$ and the nonsmooth boundary .When Hopf bifurcation occurs at $w=-3.5$,the stable branch EB_{+1} transforms into the unstable branch EB_{+2} and a stable limit cycle LC_+ appears which finally disappears at the nonsmooth Hopf bifurcation point HB_{+2} $(-1.1,1.5)$.

4. Fast-slow dynamics and its mechanism

When there exists an order gasp between the natural frequency and the frequency of periodic excitations ,the whole coupled system may behave slow-fast motion. In this section, we set the value of Ω to 0.01 and change the threshold value denoted by MT to investigate the mechanism of the bursting oscillations under different value of the threshold .

4.1 Case one: $MT=-2$

The phase diagram on plane (x,y) and the time history of state variable are shown in the Fig.5(a) and (b) when the threshold MT is set to -2. As shown in the picture, we can found that the trajectory moves with the surface within some time. We introduce the equilibrium branches and their bifurcations with TPP to study the mechanism within this time, which is show in fig.2(c) and fig(d).

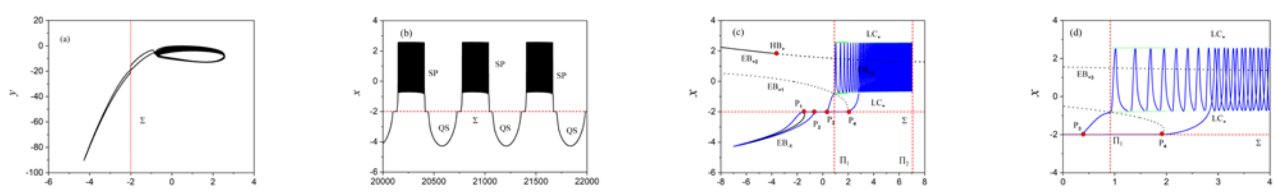

Fig. 2 $MT=-2$:(a) The phase portrait on (x,y) plane (b)Time history of x (c)The transformed phase portrait together with the bifurcation diagrams of x on (w,x) plane (d)Locally enlarged part

Let us assume that the motion start from the minimum value of the amplitude when $w=-7$. The trajectory moves to the right along the stable equilibrium branch EB_{-1} until reaches at point FB_{-1} where the fold bifurcation occurs. Then the stable branch disappears, which leading the trajectory jump up from the interface and moves into the surface at point P_2 .Then the trajectory will slide along the surface and leave it at point P_4 .This movement process constitutes a quiescentt state with a sliding structure in the time history graph. When $w=1.87$, a nonsmooth fold bifurcation occurs. The trajectory leave from the surface and attracted by the stable limit cycle LC_+ .With the increasing of the time, the trajectory begins to oscillate to the left along the limit cycle. When reach at the cross-section Π_1, the limit cycle disappears via a non-smooth homoclinic bifurcation, the trajectory jumps to the interface and enters the interface at the point P_3. This movement forms the spiking state SP in the time history graph. The trajectory may turn to the left and slide along the surface until arrive at point P_1. Because of being attracted by the stable equilibrium branch EB_{-1} ,the trajectory will jump from the surface and then move along the branch and finally go back to the beginning. Then a periodic motion complete.

4.2 Case two: $MT = -0.5$

Fig5.3(a)-(b) show the phase diagram and time history of state variable x at $MT = -0.5$. As shown in the picture that the trajectory has a part which slide along the surface. From the view of the time history, there exists a spiking state SP and quiescent state QS which keep touch with the nonsmooth surface, which is significantly different from the situation in case one. In order to further discuss the fast-slow dynamics of the system at this time, the transition phase diagram and the bifurcation diagram are superimposed as shown in Figure 5.3(c)-(d).

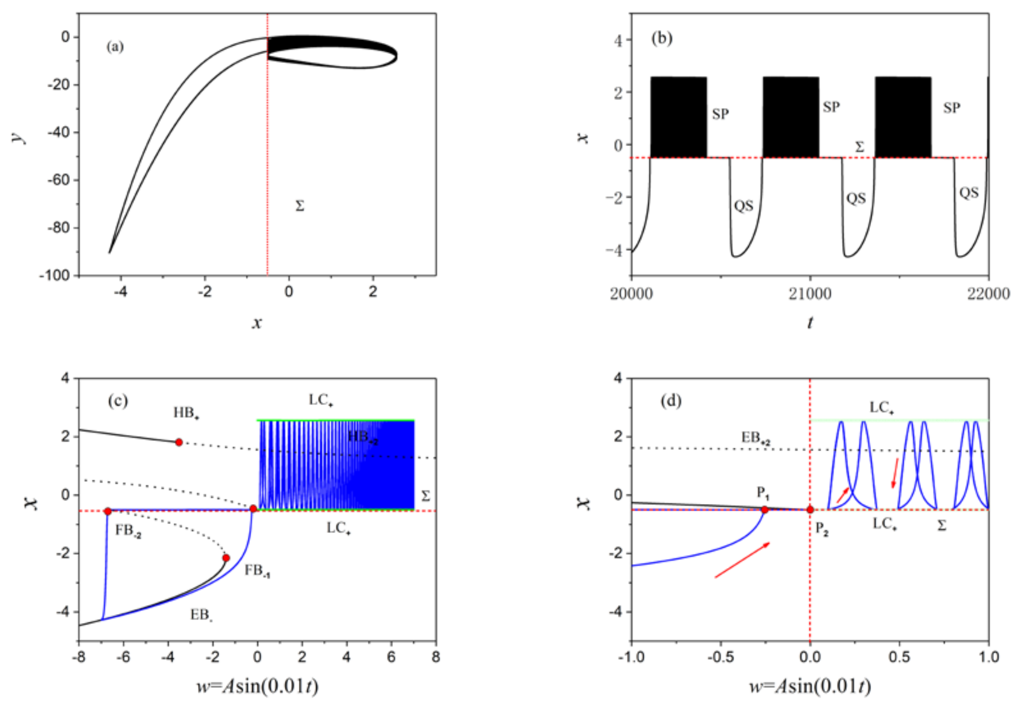

Fig. 3: $MT = -0.5$:(a) The phase portrait on (x, y) plane (b)Time history of x (c)The transformed phase portrait together with the bifurcation diagrams of x on (w, x) plane (d)Locally enlarged part of (c).

It is assumed that the trajectory starts from the point where the excitation amplitude is the smallest. Then the trajectory moves to the right along the stable equilibrium branch EB_{-1} until arrives at the point FB_{-1} where the fold bifurcation occurs. Due to the influence of the fold bifurcation, the trajectory will jump upward. It can be clearly seen from the partial enlarged view that the trajectory enters the non-smooth interface at the point of the jumping phase and keeps sliding for a period of time, which forms a quiescent state QS in the time history graph. When the trajectory moves at the interface when the excitation $w = 0$, i.e. point P , a non-smooth homoclinic bifurcation occurs, resulting in a stable limit cycle appear in the fast subsystem. The limit cycle at this time possess a non-smooth structure which slide on the interface. Due to the affection of the fast subsystem, the trajectory enters into a spiking state SP at point P_2. After that, the trajectory oscillates around a stable limit cycle LC_+ and maintains a sliding structure. When the excitation amplitude reaches to $+7$, as the interval continues to increase, the trajectory begins to oscillate and move to the left until it reaches the point P_2. This movement process corresponds to the spiking state SP in the time history graph. Due to the occurrence of non-smooth homoclinic bifurcation at the point P_2, the trajectory exits the spiking state and enters the interface and then keeps sliding along the surface to the left. When the amplitude of the external excitation is reduced to $w=-6.5$, the non-smooth fold bifurcation occurs and will cause the trajectory to jump downward, and finally return to the starting point to complete a movement cycle

4.3 Case one: $MT=-1.5$

In this case we set the thresholds value to 1.5. Fig.4 (a) and (b) represent to the phase diagram and the time history respectively. Form the picture we can find that the trajectory has a part of sliding along the nonsmooth boundary which behave the spiking state SP and the quiescent state QS in the fig.4(b). In order to further investigate the slow-fast dynamics and its mechanism of the current system, we introduce the transformed phase diagram and the equilibrium branches with bifurcation graphic of the fast subsystem which are shown in the fig4(c). Fig.4(d) is the locally enlarged part of the fig.4(c).

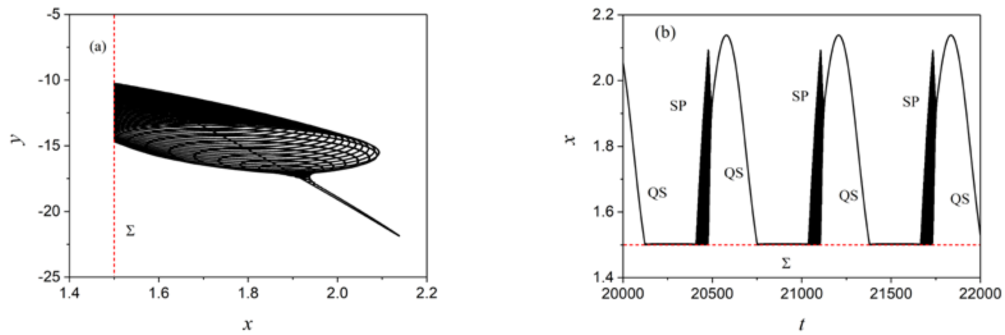

Fig. 4 $MT=1.5$ (a) The phase portrait on (x,y) plane; (b)Time history of x

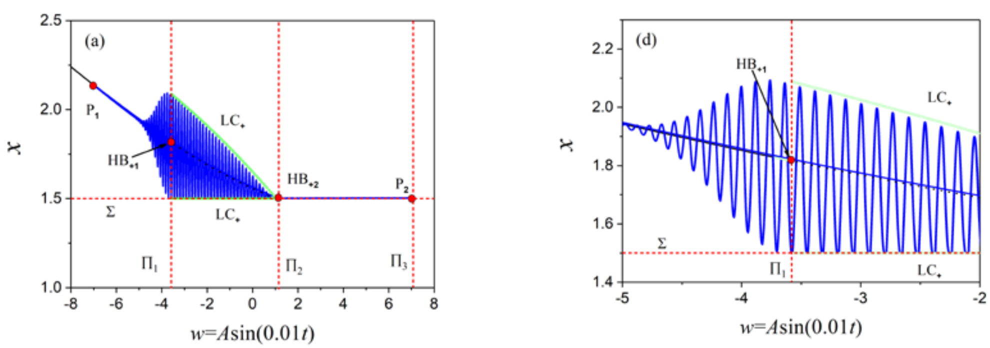

Fig. 5 (a)The transformed phase portrait together with the bifurcation diagrams of x on (w,x) plane; (b)Locally enlarged part of (a).

Without loss of generality, assuming that the trajectory start from the smallest excitation amplitude $w=-7$,i.e.the point P_1. Then the trajectory moves along with the stable equilibrium branch EB_{+1} until reaches to the point HB_{+1} where the subcritical Hopf bifurcation occurs. Due to the delay effect of the Hopf bifurcation, the trajectory move along with the unstable equilibrium branch EB_{+2} till arrive at the nonsmooth Hopf bifurcation point HB_{+2}.The trajectory crosses the bifurcation point and runs into the surface then slides along the boundary until the amplitude gets the maximum value for $w=7$,i.e. point P_2.With the increase of time ,the trajectory starts to move alone the nonsmooth surface and reaches to point HB_{+2}.The whole motion form the quiescent state in fig4(b) denoted by QS. When the slow-varying parameter cross the nonsmooth bifurcation HB_{+2},there exist an limit cycle with sliding structure which may cause the trajectory oscillate with it .When the trajectory arrive at cross-section Π_1 ,a subcritical Hopf bifurcation occurs which may cause the stable limit cycle disappear and the unstable equilibrium branch appear. However, due to the delay effect of Hopf bifurcation the trajectory may not stop oscillating at cross-section Π_1 and keep oscillating around the branch EB_{+1} ,the whole motion forms the spiking state SP .Hereafter, the trajectory moves along the equilibrium branch and eventually run back to the beginning.

4. Conclusions

In this paper we proposed a 3D Filippov system of Hindmarsh-Rose model with threshold policy control under slow-varying periodic external excitation and parametric excitation. The proposed model has the following roles. When the frequency of excitation is far smaller than the native frequency, the system may involve slow-fast dynamics. Besides, the change of the threshold may cause the motion of the nonsmooth boundary, which will cause both the different convensional and unconvensional bifurcation involved in the system, such as nonsmooth homoclinic bifurcation and nonsmooth fold bifurcation. The nonsmooth homoclinic bifurcation may cause the sliding limit cycles appear in the system which demonstrated that the system can generate sliding bursting oscillation. We also found that the delay effect may occur when the trajectory moves across the subcritical Hopf bifurcation, which may cause the trajectory oscillates around the stable equilibrium branch.

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