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Solution of Boussinesq's Equation for Infiltration Phenomenon in Horizontal Direction by Differential Transform Method

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Abstract

This paper addresses the nonlinear problem that arises in the phenomenon of infiltration in unsaturated soil. With suitable conditions, the infiltration phenomenon in unsaturated soil is defined by a second ordered partial differential equation, which gives soil moisture content. The moisture content has been observed and can be analysed using the Differential transform method. We have applied DTM method on groundwater infiltration phenomenon. The analysis is also discussed, as well as its numerical and graphical representation. We have used MATLAB for that.

Keywords- Moisture content, Differential Transform Method, Infiltration phenomenon, MATLAB, Non-linear partial differential equation

1. Introduction

Water found under the ground surface in soil pore spaces and rock formation fractures is known as groundwater. Groundwater can be recharged by natural rains, but natural discharge is more common at springs and seeps, where it can form oases or wetlands. The construction and operation of extraction wells is commonly used to extract groundwater for agricultural, urban, and industrial purposes. Evaporation, condensation, precipitation, erosion, infiltration, transpiration, and groundwater flow are all examples of how water moves from one reservoir to another. Groundwater makes up more than 90% of the liquid fresh water available on or near the earth's surface. Water infiltration is necessary for controlling salinity in water, water pollution, and agricultural purposes. The mechanism by which water on the ground surface reaches unsaturated soil or an oil reservoir is known as infiltration. In soil science, infiltration rate is an indicator of how quickly soil absorbs rainfall or irrigation. It is measured in millimetres per hour (mm/h) or inches per hour (inches/h). As the soil becomes saturated, the rate slows. An infiltrometer can be used to calculate the rate of infiltration. Gravity and capillary motion are the two forces that control infiltration. Although smaller pores are more resistant to gravity, very small pores pull water by capillary action in addition to, and even against, the force of gravity. Soil characteristics such as ease of entry, storage capacity, and transmission rate through the soil all influence the rate of infiltration. Controlling infiltration rate and capability is influenced by soil texture and structure, vegetation types and cover, soil water content, soil temperature, and rainfall intensity. The amount of water contained in a material, such as soil, rock, ceramics, fruit, or wood, is referred to as its moisture content. Water content is expressed as a ratio varying from 0 (completely dry) to the amount of the material's porosity at saturation in a range of science and technical fields. It is accessible in volumetric or mass (gravimetric) form.

In this phenomenon, water is filtered horizontally into a homogeneous unsaturated porous medium through its vertical permeable side, as shown in fig (1). It is assumed that the bottom is impermeable and that infiltrated water can only flow horizontally. The mathematical formulation results in a nonlinear partial differential equation known as Boussinesq's equation. The aim of this research is to present a physically meaningful technique for determining the effective height of the water table as a measure of a basin's initial storage capacity. Thus, using Boussinesq's theorem, an equation for mean water table height is first derived on the basis of hydraulic theory of ground water, and then the solution is obtained using the differential transform method. Darcy's law was used to measure the air pressure in the dry area and the velocity of infiltrated ground water using the height of the free surface. Let us go through some of the early history, prior to the mathematical analysis, to the extent that it has been uncovered. J. Boussinesq (1903/04)[14], a French physicist, seems to have been the first to suggest the porous medium equation $\partial_i u = \Delta(u^m), m > 1$as a mathematical model for a physical method that precisely calculates the height of the water mound in groundwater infiltration. He used the simple flow rule suggested by H.Darcy (1856) and the Dupuit (2004)[15] assumption of limited gradient. It is worth noting that the exponent is m = 2. Philip [22] described the development of infiltration equation and its numerical solution. Srivastava and Yeh [25] described one dimensional, vertical infiltration toward the water table in homogeneous and layered soils by analytical solutions. Witelski [28] expanded the Boltzmann similarity solution's applicability by adding a time-shift constant to explain the long-term action of absorption into slightly damp soil layers. Wojnar [29] addressed the Boussinesq equation for flow in a time-dependent porosity aquifer. Borana et al. [17] used the finite difference approach to achieve a numerical solution to the Boussinesq equation that arises in the one-dimensional infiltration phenomenon. Patel et al. [19] explored a homotopy analysis approach for solving Boussinesq's equation for penetration in unsaturated porous media. Chavan and Panchal [31] addressed the solution of the porous medium equation occurring in fluid flow through porous media using the homotopy perturbation approach with the Elzaki transform. Parikh [21] addressed the differential quadrature method for numerically solving Boussinesq's equation in groundwater infiltration phenomena. Desai [19] found a similarity solution to the nonlinear Boussinesq's equation that arises in the penetration of an incompressible fluid flow. Patel and Desai[30] addressed homotopy analysis method to solve the phenomena. We have applied Differential transform method.

2. Mathematical formulation

The reservoir field contains water of height OA= $h_{\max}$ = maximum height, has an impermeable bottom, and is surrounded by unsaturated homogeneous soil. The phenomenon of infiltration is well illustrated in Figure 1, which depicts a cross section of a reservoir surrounded by unsaturated porous medium.

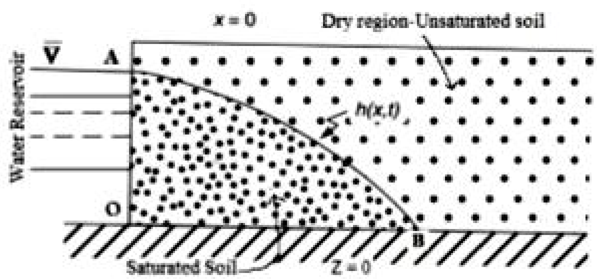

Fig 1: A scheme of groundwater Infiltration.

When OB = x = 1, the free surface height is zero, the dotted arc below the curve is saturated by infiltrated groundwater, and the dry area of unsaturated soil is above the curve. Since the bottom is presumed to be impermeable, water cannot flow downward. Infiltration is the mechanism by which reservoir groundwater enters unsaturated soil through a vertical permeable wall. The velocity of groundwater infiltration in unsaturated soil decreases as the soil becomes saturated. The infiltrated groundwater will penetrate unsaturated soil and form a curve between the saturated and unsaturated soils, which is known as a water table or water mound. We made several hypotheses to better explain this one-dimensional infiltration phenomenon and to help with its mathematical formulation. Boussinesq's equation [3] is the governing nonlinear partial differential equation for the height of infiltrated water. The aim of the infiltration analysis is to determine the effective height of the free surface as a measure of a porous stratum's initial storage ability.

With the following simplification assumptions, groundwater is infiltrated through the vertical adjacent side:

1. The stratum is $h_{\max}$ in height and rests on top of a horizontal impermeable bed (z=0);
2. The transversal variable y is ignored.
3. The water mass that infiltrates the soil is located in a region known as

□ □□□x, z □□R : z □h □x, t □□

We're presuming that there isn't a partial saturation region. This is a model of evolution. Clearly, 0 □h □x, t □□ $h_{\max}$, $h_{\max}$ For this problem, the maximum height of the free surface and the free boundary function 'h' are both unknown functions. In this case, we have a system of three equations in a variable domain, with the two velocity components 'u', 'w', and 'p' as unknowns: one equation of mass conservation for an incompressible fluid and two Navier-Stokes type equations for momentum conservation.

The resulting system is overly complicated and can be simplified for practical computation by introducing a suitable assumption, the hypothesis of almost horizontal flow, which states that an almost horizontal flow with speed u □u, 0□has small gradients. As a result, the vertical component of the momentum equation can be written as follows:

$$p\left(\frac{du_z}{dt} + u.\Delta u_z\right) = -\frac{\partial p}{\partial z} - \rho g \qquad (1)$$

The term inertial has been omitted (the left-hand side). For the first approximation, integration equation (1) with respect to z gives

p □□gz □constant (2)

We will now compute the free surface constant *z* □*h* □*x*, *t*□ If we apply pressure continuity across the interface (assuming constant atmospheric pressure in the air that fills the pores of the dry zone), we obtain p=0. z > h(x, t)). Then we get

p □□*g* □*h* □z□
□□□(3)

As a result, the pressure can be calculated using the hydrostatic approximation. We now proceed to the mass conservation law, which will provide us with the equation. We proceed in the following manner: Let us take a gander at a section.

S □□*x*, *x* □*a* □□□0,*C*□ Then

$$\emptyset \frac{\partial}{\partial t}\int_x^{x+a}\int_0^h dydx = -\int_{\partial s} u.ndl \qquad (4)$$

Where ' □' is the medium's porosity, i.e., the proportion of volume available for flow circulation, and 'u' is the seepage velocity, which obeys Darcy's law in the form that includes gravity effects,

$$u = -\frac{k}{\mu}\Delta(\mathrm{p} + \rho g z) \qquad (5)$$

We have $u.\mathrm{n} \approx (u,0).(1,0) = u$ i.e.$\left(\frac{k}{\mu}\right)p_x$ on the right-hand lateral surface

while the velocity is shown as '-u' on the left. Using the 'p' formula and differentiating with respect to 'x,' we obtain

$$\emptyset \frac{\partial h}{\partial t} = \frac{pgK}{\mu}\frac{\partial}{\partial x}\int_0^h \frac{\partial}{\partial x} hdz \qquad (6)$$

Finally we obtained Boussinesq's equation as

$$\frac{\partial h}{\partial t} = \frac{pgK}{2\mu\emptyset}\frac{\partial^2}{\partial x^2}(h^2) \qquad (7)$$

With constant $\beta = \frac{pgK}{2\mu\emptyset}$, this is the fundamental equation in ground water infiltration that reflects the porous medium equation. Considering a new dimensionless variable

$T = \frac{pgK}{\mu\emptyset}t$ and $X = \frac{x}{L}$ in Equation (7) , it gives,

$$\frac{\partial h}{\partial T} = \left\{h\frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2\right\} \qquad (8)$$

The height of the water mound is given by equation (8) with initial and boundary conditions as,

h(0,T)=h_{max} X=0 and T> 0 (9)

which satisfied the initial condition

h(X,0)=h_0 X>0 and T=0 (10)

Equation (8) is an essential equation in groundwater infiltration.

3. Differential Transform Method

Zhou[1] was the first to propose the DTM, which solved linear and nonlinear issues in electrical circuit problems. This methodology was established by Chen and Ho[2] for partial differential equations, and Ayaz[13] extended it to the differential equation approach. In recent years, several researchers have employed this methodology to solve other types of equations.

3.1 Two dimensional Differential Transform Method

Consider a function of two variables *s(x, t)* and suppose that it can be represented as product of two single-variable functions, i.e., s(x, t) = *f*(*x*)*g*(*t*)*:* Based on the properties of differential transform , function *s(x, t)* can be represented as

$$s(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} S(i,j)x^i t^j \,, \qquad (11)$$

Where $S(i,j)$ = F(i)G(j) is called the spectrum of s(x, t).

If function *s(x, t)* is analytic and differentiated continuously with respect to time *t* and space *x* in the domain of interest, then let

$$S(m\,,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n}s(x,t)\right]_{\substack{x=0\\t=0}} \qquad (12)$$

where the t-dimensional spectrum function $S(m,n)$ is the transformed function which is called T-function in brief. The differential inverse transform of $S(m,n)$ is defined as follows:

$$s(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S(m,n)x^m t^n \quad (13)$$

Combining (12) and (13) gives the solution as

$$s(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n} s(x,t)\right]_{\substack{x=0\\t=0}} x^m t^n \quad (14)$$

Function Form	Transformed Form
$s(x,t) = u(x,t) \pm v(x,t)$	$S(m,n) = U(m,n) \pm V(m,n)$
$s(x,t) = cu(x,t)$	$S(m,n) = cU(m,n)$
$s(x,t) = \frac{\partial}{\partial x}u(x,t)$	$S(m,n) = (m+1)U(m+1,n)$
$s(x,t) = \frac{\partial}{\partial t}u(x,t)$	$S(m,n) = (n+1)U(m,n+1)$
$s(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$	$S(m,n) = \frac{(m+r)!\,(n+s)!}{m!\,n!}U(m+r,n+s)$
$s(x,t) = u(x,t)v(x,t)$	$S(m,n) = \sum_{r=0}^{m}\sum_{s=0}^{n} U(r,n-s)V(m-r,s)$
$s(x,t) = x^\alpha t^\beta$	$S(m,n) = \delta(m-\alpha, n-\beta) = \begin{cases}1 & m=\alpha, n=\beta\\ 0 & otherwise\end{cases}$

Table 1

4. Solution of groundwater Infiltration Phenomenon in Horizontal Direction

According to the DTM and table 1, we can construct the following transformation of Equation (8) $\frac{\partial h}{\partial T} = \left\{h\frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2\right\}$

With Initial condition $h(X,0) = 0.8e^{-X}$ as:

$$\begin{aligned}(n+1)H(m,n+1) &= \sum_{r=0}^{m}\sum_{s=0}^{n}(r+1)(m-r+1)H(r+1,n-s)H(m-r+1,s)\\ &+ \sum_{r=0}^{m}\sum_{s=0}^{n}(m-r+1)(m-r+2)H(r,n-s)H(m-r+2,s)\end{aligned} \quad (15)$$

By definition of DTM

$$H(m,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial X^m \partial T^n} h(X,T)\right]_{\substack{X=0\\t=0}} \quad (16)$$

To Apply initial condition in equation (16) putting n=T=0 we get,

$H(m\,,0) = \frac{1}{m!0!}\left[\frac{\partial^m}{\partial X^m}h(X,0)\right]_{X=0}$ (17)

Applying initial condition in equation (17) and putting different values of m , i.e m=0,1,2,3,4,5,… we get

$$H(0\,,0) = \frac{1}{0!}\left[\frac{\partial^0}{\partial X^0}h(X,0)\right]_{X=0}$$

$$H(0\,,0) = \frac{1}{1}[0.8\mathrm{e}^{-\mathrm{X}}]_{X=0}$$

So we get H(0,0) =0.8

$$H(1\,,0) = \frac{1}{1!}\left[\frac{\partial^1}{\partial X^1}h(X,0)\right]_{X=0}$$

$$H(1\,,0) = \frac{1}{1}\left[\frac{\partial^1}{\partial X^1}0.8\mathrm{e}^{-\mathrm{X}}\right]_{X=0}$$

$$H(1\,,0) = [-0.8\mathrm{e}^{-\mathrm{X}}]_{X=0}$$

So we get H(1,0) = -0.8

$$H(2\,,0) = \frac{1}{2!}\left[\frac{\partial^2}{\partial X^2}h(X,0)\right]_{X=0}$$

$$H(2\,,0) = \frac{1}{2}\left[\frac{\partial^2}{\partial X^2}0.8\mathrm{e}^{-\mathrm{X}}\right]_{X=0}$$

$$H(2\,,0) = \frac{1}{2}[0.8\mathrm{e}^{-\mathrm{X}}]_{X=0}$$

So we get H(2,0) = $\frac{0.8}{2}$

In similar manner, we will get H(3,0) = $-\frac{0.8}{6}$

H(4,0) = $\frac{0.8}{24}$

H(5,0) = $-\frac{0.8}{120}$

Therefore , the values that we found are

H(0,0) =0.8 , H(1,0) = -0.8 , H(2,0) = $\frac{0.8}{2}$, H(3,0) = $-\frac{0.8}{6}$, H(4,0) = $\frac{0.8}{24}$, H(5,0) = $-\frac{0.8}{120}$ (18)

Putting different values of m and n in equation (15) we got different H(X,T) which are as follows:

H(0,0) = 0.8	H(1,0) = -0.8	H(2,0) = 0.4	H(3,0) = −0.133
H(0,1) = 1.28	H(1,1) = -2.56	H(2,1) = 2.56	H(3,1) = −1.71
H(0,2) = 4.608	H(1,2) = -13.824	H(2,2) =20.736	H(3,2) = −20.74
H(0,3) = 23.12	H(1,3) = −92.5	H(2,3) = 184.96	H(3,3) = -246.6

And so on…

Now by equation (13) we get,

$h(X,T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}H(m,n)X^mT^n$

so, $h(X,T) = H(0,0)X^0T^0 + H(1,0)X^1T^0 + H(2,0)X^2T^0 + H(3,0)X^3T^0 + H(0,1)X^0T^1 + H(1,1)X^1T^1 + H(2,1)X^2T^1 + H(3,1)X^3T^1 + H(0,2)X^0T^2 + H(1,2)X^1T^2 + H(2,2)X^2T^2 + H(3,2)X^3T^2 + H(0,3)X^0T^3 + H(1,3)X^1T^3 + H(2,3)X^2T^3 + H(3,3)X^3T^3 + \cdots$

From above table and values of X and T we have,

$$h(x,t) = 0.8 - 0.8x + 0.4x^2 - 0.133x^3 - 0.128t + 0.256xt - 0.256x^2t + 0.171x^3t + 0.04608t^2 - 0.13824xt^2 + 0.20736x^2t^2 - 0.2074x^3t^2 - 0.02312t^3 + 0.0925xt^3 - 0.18496x^2t^3 + 0.2466x^3t^3 + \cdots \quad (19)$$

Where $T = \frac{pgK}{\mu\emptyset}t$ and $X = \frac{x}{L}$.

4.1 Numerical and Graphical presentation

MATLAB coding was used to generate numerical and graphical representations of Equation (19). The graph of height h vs. x for fixed time t=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0 is shown in Figure 2. The table below shows the numerical values for height for various distances x at fixed times t= 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. The values of the parameters were taken into consideration from standard literature in this case as g=-9.8, p=0.02, $\emptyset = 0.218$, L=1.

	h(x,t)									
x	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5	t=0.6	t=0.7	t=0.8	t=0.9	t=1.0
0.1	0.7137	0.7042	0.6951	0.6865	0.6781	0.6699	0.6619	0.6546	0.6459	0.6378
0.2	0.6467	0.6388	0.6312	0.6242	0.6175	0.6111	0.6049	0.5989	0.5932	0.5875
0.3	0.5859	0.5793	0.5731	0.5671	0.5613	0.5557	0.5501	0.5447	0.5393	0.5339
0.4	0.5306	0.5252	0.5201	0.5151	0.5103	0.5057	0.5011	0.4967	0.4923	0.4879
0.5	0.4806	0.4762	0.4719	0.4678	0.4639	0.4600	0.4563	0.4526	0.4490	0.4455
0.6	0.4352	0.4316	0.4280	0.4246	0.4217	0.4180	0.4147	0.4115	0.4083	0.4051
0.7	0.3942	0.3912	0.3883	0.3855	0.3828	0.3801	0.3775	0.3749	0.3724	0.3699
0.8	0.3569	0.3544	0.3520	0.3497	0.3474	0.3451	0.3430	0.3409	0.3388	0.3368
0.9	0.3232	0.3212	0.3192	0.3173	0.3154	0.3136	0.3118	0.3100	0.3083	0.3066
1.0	0.2926	0.2909	0.2893	0.2877	0.2862	0.2846	0.2832	0.2817	0.2803	0.2796

Table 2

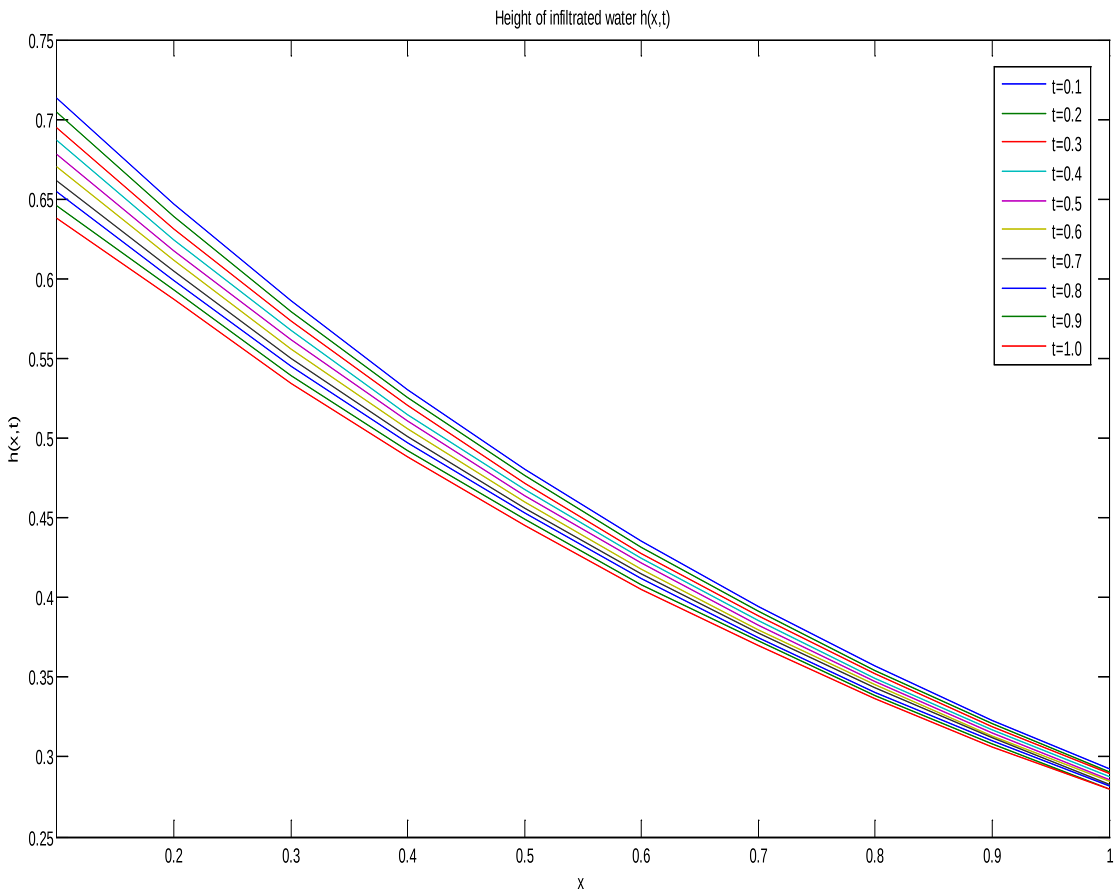

Fig 2 : Height of Infiltrated Water

5. Conclusion

We discussed the infiltration phenomena in unsaturated porous media, as well as its mathematical formulation, which leads to the Boussinesq equation. The Differential transform Method is used to find the solution to the Boussinesq problem. Equation (19) shows the solution by differential transform method. When length x and duration t both increase, the solution h(x,t) or moisture contain decreases. We concluded that the height of infiltrated groundwater decreases when length increases, so does the time, which goes with its physical explanation.

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