IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 8 Issue 6, June 2021
ISSN (Online) 2348 – 7968 | Impact Factor (2020) – 6.72
www.ijiset.com

Towards The Efficiency of the Ratio Estimator for Population Median in Survey Sampling

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ABSTRACT:

This study leveraged the fact that researchers in survey sampling sometimes do not take into consideration the tool that will be most appropriate in the measure of location. As a result, users of statistics often go for the mean or total, which has been widely discussed in the finite population sampling literature, unlike the median, which is more complicated to deal with given that it has to do with ordered data. This study suggests an estimator of population median in single and double sampling technique with two auxiliary variables. From the result obtained, it suffices to say that the proposed estimators perform better in terms of gain in efficiency when the considered variables are from a highly skewed distribution, such as income, expenditure, scores. In addition, it is evident to say that the proposed estimators were less biased and more precise than the existing estimators of their class.

1.0 **Introduction**

Most times in survey sampling, some researchers do not consider the tool that will be most appropriate in the measure of location. As a result, users of Statistics often go for the mean or total, which has been widely discussed in the finite population sampling literature, unlike the median, which is more complicated to deal with since it has to do with ordered data. However, the median, unlike the mean, performs better when the considered variables are from a highly skewed distribution. In surveys involving the estimation of income, expenditure, scores, etc., it is very reasonable to assume that the population median unlike the population mean, is known, hence the possibility of incorporating auxiliary information in the formulation of such estimators.

Authors like Gross (1980), Kuk and Mak (1989), Singh et al. (2003), Singh and Solanki (2013), Aladag and Cingi (2015) have contributed in the estimation of population median. Works by Enang et.al, (2016) on the alternative exponential median estimator, Shabbir and Gupta (2017) on a generalized difference type estimator for population median, and Iseh (2020) on enhancing the efficiency of ratio estimator of population median by calibration techniques were added advantage in this area. However, more research is required in estimating population median. On further improvement of the median estimator, Singh, Joarder, and Tracy (2001) extended the ratio estimator to two-phase sampling, while Singh, Singh, and Upadhyaya (2007) suggested the ratio-type estimator in two-phase sampling using two auxiliary variables. In addition, Jhajj, Kaur, and Jhajj (2016) defined a ratio exponential-type estimator in two-phase sampling with two auxiliary variables. On the lines of Shabbir and Gupta (2017), Baig, Masood, and Tarray (2019) suggested an improved class of difference-type estimators for population median using two auxiliary variables under a simple random scheme and two-phase sampling scheme.

So far, the shortcoming of the existing estimators is that, while some of these estimators are less biased with large mean square error (MSE), others are highly biased with less MSE. Based on these developments as a benchmark, this study proposes a separate-ratio-product exponential-type estimator in simple random and two-phase sampling schemes with two auxiliary variables that will be more precise with greater gains in efficiency to estimate the finite population.

1.1 Notations

Consider a finite population $U = \{u_1, u_2, \ldots, u_N\}$ with size N. Let Y, X, and Z be the study and auxiliary variables respectively. Let y_i represents the samples of the interest variable, x_i and z_i represent samples of the auxiliary variables known for every unit in the population for the i^{th} element drawn under SRSWOR. Let $f_Y(M_Y)$, $f_X(M_X)$, and $f_Z(M_Z)$ represent the density functions of the random variables with $\widehat{M}_y$, $\widehat{M}_x$, and $\widehat{M}_z$ being the samples from the population median M_Y, M_X, and M_Z respectively, with correlation coefficient $\rho_{M_Y M_X} = 4(P_{11} - 0.25)$, where $P_{11} = P(Y \leq M_Y \cap X \leq M_X)$, $\rho_{M_Y M_Z} = 4(P_{11} - 0.25)$, where $P_{11} = P(Y \leq M_Y \cap Z \leq M_Z)$, and $\rho_{M_X M_Z} = 4(P_{11} - 0.25)$, where $P_{11} = P(X \leq M_X \cap Z \leq M_Z)$, (considering the continuous distributions of all variables $y, x,$ and x with their marginal densities respectively as $N \to \infty$).

For large sample approximations, the following are obtainable:

$$\widehat{M}_y = M_Y(1 + e_0), \qquad \widehat{M}_x = M_X(1 + e_1), \qquad \widehat{M}_z = M_Z(1 + e_2),$$

$$e_0 = \frac{\widehat{M}_y - M_Y}{M_Y} \qquad e_1 = \frac{\widehat{M}_x - M_X}{M_X} \qquad e_2 = \frac{\widehat{M}_z - M_Z}{M_Z}$$

$$E(e_0) = E(e_1) = E(e_2) = 0 \qquad \lambda = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{N}\right) \qquad \lambda_1 = \frac{1}{4}\left(\frac{1}{m} - \frac{1}{N}\right)$$

$E(e_0^2) = \lambda C_{M_Y}^2$ $\quad E(e_1^2) = \lambda C_{M_X}^2$ $\quad E(e_2^2) = \lambda C_{M_Z}^2$

$E(e_0 e_1) = \lambda C_{M_Y} C_{M_X} \rho_{M_Y M_X}$ $\quad E(e_0 e_2) = \lambda C_{M_Y} C_{M_Z} \rho_{M_Y M_Z}$

$C_{M_Y} = \{M_Y f_Y(M_Y)\}^{-1}$ $\quad C_{M_X} = \{M_X f_X(M_X)\}^{-1}$ $\quad C_{M_{Xh}} = \{M_X f_X(M_X)\}^{-1}$

$k_1 = \frac{C_{M_Y} \rho_{M_Y M_X}}{C_{M_X}}$ $\quad k_2 = \frac{C_{M_Z} \rho_{M_X M_Z}}{C_{M_X}}$

Where, it is also assumed that the distribution function $f_Y(M_Y)$, $f_X(M_X)$, and $f_X(M_Z)$ are nonnegative.

2.0 Existing Estimators under Simple Random Sampling

This section considers some existing estimators in simple random sampling in estimating population median and the expression of bias and MSE up to the first-order approximation as follows;

A: The median estimator (per unit) due to Gross (1980) is given by

$$\widehat{M}_G = \widehat{M}_y$$

$$var(\widehat{M}_G) = \lambda M_Y^2 C_{M_Y}^2 \tag{1}$$

B: The median estimator (classical ratio) by Kuk and Mak (1989) is given by

$$\widehat{M}_R = \widehat{M}_y \left(\frac{M_X}{\widehat{M}_x}\right)$$

$$B(\widehat{M}_R) = \lambda M_Y C_{M_X}^2 (1 - k_1)$$

$$MSE(\widehat{M}_R) = \lambda M_Y^2 \left[C_{M_Y}^2 + C_{M_X}^2 (1 - 2k_1)\right] \tag{2}$$

C: The median estimator (exponential ratio) following Bahl and Tuteja (1991) is given by

$$\widehat{M}_{ER} = \widehat{M}_y exp\left[\frac{M_X - \widehat{M}_x}{M_X + \widehat{M}_x}\right]$$

$$B(\widehat{M}_{ER}) = \frac{\lambda M_Y C_{M_X}^2 (3 - 4k_1)}{8}$$

$$MSE(\widehat{M}_{ER}) = \lambda M_Y^2 \left[C_{M_Y}^2 + \frac{C_{M_X}^2}{4}(1 - 4k_1)\right] \tag{3}$$

D: The median estimator (chain ratio-type) by Kadilar and Cingi (2003) is given by

IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 8 Issue 6, June 2021
ISSN (Online) 2348 – 7968 | Impact Factor (2020) – 6.72
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$$\widehat{M}_{CR} = \widehat{M}_y \left(\frac{M_X}{\widehat{M}_x}\right)^2$$

$$B\left(\widehat{M}_{CR}\right) = \lambda M_Y C_{M_X}^2 (1 + 2k_1)$$

$$MSE\left(\widehat{M}_{CR}\right) = \lambda M_Y^2 \left[C_{M_Y}^2 + 4C_{M_X}^2 (1 + k_1)\right] \quad (4)$$

E: The median estimator (product-type) following Robson (1957) and Murthy (1964) is given by

$$\widehat{M}_P = \widehat{M}_y \left(\frac{\widehat{M}_x}{M_X}\right)$$

$$B\left(\widehat{M}_P\right) = \lambda M_Y C_{M_X}^2 k_1$$

$$MSE\left(\widehat{M}_P\right) = \lambda M_Y^2 \left[C_{M_Y}^2 + C_{M_X}^2 (1 + 2k_1)\right] \quad (5)$$

F: The median estimator (exponential product-type) following Bahl and Tuteja (1991) is given by

$$\widehat{M}_{EP} = \widehat{M}_y exp\left[\frac{\widehat{M}_x - M_X}{M_X + \widehat{M}_x}\right]$$

$$B\left(\widehat{M}_{EP}\right) = \frac{\lambda M_Y C_{M_X}^2 (4k_1 - 1)}{8}$$

$$MSE\left(\widehat{M}_{EP}\right) = \lambda M_Y^2 \left[C_{M_Y}^2 + \frac{C_{M_X}^2}{4}(1 + 4k_1)\right] \quad (6)$$

G: The median estimator (alternative exponential) due to Enang et al. (2016) is given by

$$\widehat{M}_{AE} = \alpha_1 \left[\widehat{M}_y exp\left[\frac{M_X - \widehat{M}_x}{M_X + \widehat{M}_x}\right]\right] + \alpha_2 \left[\widehat{M}_y exp\left[\frac{\widehat{M}_x - M_X}{M_X + \widehat{M}_x}\right]\right]$$

$$B\left(\widehat{M}_{AE}\right) = \lambda M_Y C_{M_X}^2 (4k_1 - 8{k_1}^2 + 1)$$

$$MSE\left(\widehat{M}_{AE}\right) = \lambda M_Y^2 C_{M_Y}^2 (1 - {\rho_{M_Y M_X}}^2) \quad (7)$$

H: Shabbir and Gupta (2017) suggested a generalized difference-type estimator for population median as

$$\widehat{M}_{PP}^G = \left[m_1 \widehat{M}_y + m_2\left(M_X - \widehat{M}_x\right)\right]\left[\left(\frac{aM_X + b}{a\widehat{M}_x + b}\right) exp\left\{\frac{\alpha_2 a(M_X - \widehat{M}_x)}{a\{(\gamma - 1)M_X + \widehat{M}_x\} + 2b}\right\}\right]$$

where m_1 and m_2 are unknown constants whose values are to be determined.
Let a and b are defined to be unknown population parameters, and α_1, α_2 and γ are scalar quantities which can take different values like $\alpha_1 = b = 0$ and $\alpha_2 = a = \gamma = 1$.

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At the optimum values of $m_{1(opt)} = \frac{1-\frac{1}{2}\lambda M_X^2}{1+\lambda M_Y^2\left(1-\rho_{M_Y M_X}{}^2\right)}$ and $m_{2(opt)} = \frac{M_Y}{M_X}\left[1 + m_{1(opt)}\left\{\frac{\rho_{M_Y M_X} C_{M_Y}}{C_{M_X}} - 2\right\}\right]$

The expressions for the bias and the mean square error up to the first order of approximation are as follows:

$$B\left(\widehat{M}_{PP}^{G}\right) \cong \left(m_{1(opt)} - 1\right)M_Y + m_{2(opt)}\left\{\lambda M_Y\left(\frac{3}{8}C_{M_X}^2 - C_{M_Y}\right) + \lambda M_X C_{M_X}^2\right\} \quad \text{and}$$

$$MSE\left(\widehat{M}_{PP}^{G}\right) \cong \frac{\lambda M_Y^2}{1+\lambda M_Y^2\left(1-\rho_{M_Y M_X}{}^2\right)}\left[C_{M_Y}^2\left(1-\rho_{M_Y M_X}{}^2\right)\left(1-\lambda C_{M_X}^2\right) - \frac{1}{4}C_{M_X}^4\right] \tag{8}$$

I: Baig, Masood, and Tarray (2019) suggested an improved class of difference-type estimators for population median using two auxiliary variables

$$\widehat{M}_P^I = \left[\widehat{M}_y + m_1\left(M_X - \widehat{M}_x\right)\right]\left[m_2 exp\left(\frac{M_Z-\widehat{M}_z}{M_Z+\widehat{M}_z}\right) + (1-m_2)exp\left(\frac{\widehat{M}_z-M_Z}{M_Z+\widehat{M}_z}\right)\right]$$

Where m_1 and m_2 are unknown constant.

$$B\left(\widehat{M}_P^I\right) = \lambda\left[m_1 M_X C_{M_{XZ}}\left(m_2 - \frac{1}{2}\right) + M_Y C_{M_{YZ}}\left(\frac{1}{2} - m_2\right)\right]M_Y$$

where $m_{1(opt)} = \frac{M_Y C_{M_Y}\left(\rho_{M_X M_Z}\rho_{M_Y M_Z} - \rho_{M_Y M_X}\right)}{M_X C_{M_X}\left(1-\rho_{M_X M_Z}^2\right)}$ and

$$m_{2(opt)} = \frac{C_{M_Z}\left(\rho_{M_X M_Z}^2 - 1\right) + 2C_{M_Y}\left(\rho_{M_X M_Z}\rho_{M_Y M_X} - \rho_{M_Y M_Z}\right)}{2C_{M_Z}\left(\rho_{M_X M_Z}^2 - 1\right)}$$

$$MSE\left(\widehat{M}_P^I\right) = \frac{\lambda M_Y^2 C_{M_Y}^2}{\left(1-\rho_{M_X M_Z}{}^2\right)}\left[\left(1 - \rho_{M_X M_Z}^2 - \rho_{M_Y M_X}^2 - \rho_{M_Y M_Z}^2 + 2\rho_{M_X M_Z}\rho_{M_Y M_X}\rho_{M_Y M_Z}\right)\right] \tag{9}$$

3.0 The Proposed Estimator under Simple Random Sampling with one Variable

$$\widehat{M}_{srs}(\alpha) = \widehat{M}_y\left[\alpha\frac{M_X}{\widehat{M}_x} + (1-\alpha)\frac{\widehat{M}_x}{M_X}\right]exp\left[\frac{(M_X-\widehat{M}_x)}{(M_X+\widehat{M}_x)}\right] \tag{10}$$

$$\widehat{M}_{srs}(\alpha) = M_Y(1+e_0)[1 + e_1 - 2\alpha e_1 + \alpha e_1^2\ ]exp\left\{\frac{-e_1}{2}\left[1 - \frac{e_1}{2} + \frac{e_1^2}{4}\right]\right\}$$

$$\widehat{M}_{srs}(\alpha) = M_Y\left[1 + e_0 + \left(\frac{1}{2} - 2\alpha\right)e_1 + \left(\frac{1}{2} - 2\alpha\right)e_0 e_1 + \left(2\alpha - \frac{1}{8}\right)e_1^2\ \right]$$

$$\widehat{M}_{srs}(\alpha) - M_Y = M_Y\left[e_0 + \left(\frac{1}{2} - 2\alpha\right)e_1 + \left(\frac{1}{2} - 2\alpha\right)e_0 e_1 + \left(2\alpha - \frac{1}{8}\right)e_1^2\ \right] \tag{11}$$

$$Bias\left(\hat{M}_{srs}(\alpha)\right) = M_Y\left[\left(2\alpha - \frac{1}{8}\right)\lambda C_{M_X}^2 + \left(\frac{1}{2} - 2\alpha\right)\lambda C_{M_Y}C_{M_X}\rho_{M_YM_X}\right] \quad (12)$$

Squaring both sides of (11), retaining terms to the second-degree and taking expectations, the MSE of $\hat{M}_{st}(\alpha)$ to the first order of approximation obtained as;

$$MSE\left(\hat{M}_{srs}(\alpha)\right) = M_Y^2\left[\lambda C_{M_Y}^2 + \left(\frac{1}{2} - 2\alpha\right)^2\lambda C_{M_X}^2 + 2\left(\frac{1}{2} - 2\alpha\right)\lambda C_{M_Y}C_{M_X}\rho_{M_YM_X}\right] \quad (13)$$

Minimizing (13) with respect to α gives

$$\alpha = \frac{2k_1+1}{4}$$

$$MSE_{opt}\left(\hat{M}_{srs}(\alpha)\right) = \lambda M_Y^2 C_{M_Y}^2(1-\rho_{M_YM_X}^2) \quad (14)$$

And the optimum bias becomes

$$Bias_{opt}\left(\hat{M}_{srs}(\alpha)\right) = \lambda M_Y\left[\frac{3}{8}C_{M_X}^2 - C_{M_Y}^2\rho_{M_YM_X}^2 + C_{M_Y}C_{M_X}\rho_{M_YM_X}\right] \quad (15)$$

3.1 Simple Random Sampling with Two Variables

$$\hat{M}_{srs}^*(\alpha) = \hat{M}_y\left[\alpha\frac{M_X}{\hat{M}_x} + (1-\alpha)\frac{\hat{M}_x}{M_X}\right]exp\left[\frac{(\hat{M}_z - M_Z)}{(M_Z+\hat{M}_z)}\right] \quad (16)$$

$$Bias\left(\hat{M}_{srs}^*(\alpha)\right) = \lambda M_Y\left[\alpha C_{M_X}^2 + \frac{3}{8}C_{M_Z}^2 + (1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_XM_Y} - (1-2\alpha)\frac{C_{M_X}C_{M_Z}\rho_{M_XM_Z}}{2} - \frac{C_{M_Y}C_{M_Z}\rho_{M_YM_Z}}{2}\right] \quad (17)$$

$$MSE\left(\hat{M}_{srs}^*(\alpha)\right) = \lambda M_Y^2\left[C_{M_Y}^2 + (1-2\alpha)^2C_{M_X}^2 + \frac{C_{M_Z}^2}{4} - (1-2\alpha)C_{M_X}C_{M_Z}\rho_{M_XM_Z} + 2(1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_XM_Y} - C_{M_Y}C_{M_Z}\rho_{M_YM_Z}\right] \quad (18)$$

Minimizing (18) with respect to α gives $\alpha = \frac{2C_{M_X}^2 + 2C_{M_X}C_{M_Y}\rho_{M_XM_Y} - C_{M_X}C_{M_Z}\rho_{M_XM_Z}}{4C_{M_X}^2}$

Substituting into (18) gives

$$MSE_{opt}\left(\hat{M}_{srs}^*(\alpha)\right) = \lambda M_Y^2\left[C_{M_Y}^2\left(1-\rho_{M_YM_X}^2\right) + \frac{C_{M_Z}^2}{4}\left(1-\rho_{M_XM_Z}^2\right) + C_{M_Y}C_{M_Z}\left(\rho_{M_XM_Z}\rho_{M_YM_X} - \rho_{M_YM_Z}\right)\right]$$

$$MSE_{opt}\left(\hat{M}_{srs}^*(\alpha)\right) = \lambda M_Y^2\left[C_{M_Y}^2 + \frac{C_{M_Z}^2}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z}\right] \quad (19)$$

And the minimum bias given as

$$Bias_{opt}\left(\widehat{M}_{srs}^{*}(\alpha)\right) = \lambda M_Y\left[\frac{C_{M_X}^2}{2} + \frac{3}{8}C_{M_Z}^2 + \frac{C_{M_X}C_{M_Y}\rho_{M_XM_Y}}{2} - \frac{C_{M_X}C_{M_Z}\rho_{M_XM_Z}}{4} - C_{M_Y}^2\rho_{M_XM_Y}^2 - \frac{C_{M_Z}^2\rho_{M_XM_Z}^2}{4} - \frac{C_{M_Y}C_{M_Z}\rho_{M_YM_Z}}{2} + C_{M_Y}C_{M_Z}\rho_{M_XM_Z}\rho_{M_XM_Y}\right] \quad (20)$$

3.2 Efficiency Comparison under Simple Random Sampling

To complement the theoretical formulations, conditions necessary for efficiency of the proposed and existing estimators are hereby considered. Emphasis is on the proposed separate ratio-product exponential estimator with two auxiliary variables, $MSE_{opt}\left(\widehat{M}_{srs}^{*}(\alpha)\right)$ with respect to the existing estimators considered in this work.

Condition i: Efficiency comparison of $\widehat{M}_{srs}^{*}(\alpha)$ with the sample median estimator $\widehat{M}_G$

By Eq. 20 and Eq. 1, $MSE_{opt}\left(\widehat{M}_{srs}^{*}(\alpha)\right) < var\left(\widehat{M}_G\right)$ if

$$\frac{C_{M_Z}^2}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < 0$$

Condition ii: Efficiency comparison of $\widehat{M}_{srs}^{*}(\alpha)$ with the sample median estimator $\widehat{M}_R$

By Eq. 20 and Eq. 2, $MSE_{opt}\left(\widehat{M}_{srs}^{*}(\alpha)\right) < MSE\left(\widehat{M}_R\right)$ if

$$\frac{C_{M_Z}^2}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < C_{M_X}^2(1 - 2k_1)$$

Condition iii: Efficiency comparison of $\widehat{M}_{srs}^{*}(\alpha)$ with the sample median estimator $\widehat{M}_{ER}$

By Eq. 20 and Eq. 3, $MSE_{opt}\left(\widehat{M}_{srs}^{*}(\alpha)\right) < MSE\left(\widehat{M}_{ER}\right)$ if

$$\frac{C_{M_Z}^2}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < \frac{C_{M_X}^2}{4}(1 - 4k_1)$$

Condition iv: Efficiency comparison of $\widehat{M}_{srs}^{*}(\alpha)$ with the sample median estimator $\widehat{M}_{CR}$

By Eq. 20 and Eq. 4, $MSE_{opt}\left(\widehat{M}_{srs}^{*}(\alpha)\right) < MSE\left(\widehat{M}_{CR}\right)$ if

$$\frac{C_{M_Z}^2}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < 4C_{M_X}^2(1 + k_1)$$

Condition v: Efficiency comparison of $\widehat{M}_{srs}^{*}(\alpha)$ with the sample median estimator $\widehat{M}_P$

By Eq. 20 and Eq. 5, $MSE_{opt}\left(\widehat{M}^*_{srs}(\alpha)\right) < MSE\left(\widehat{M}_P\right)$ if

$$\frac{C^2_{M_Z}}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < C^2_{M_X}(1 + 2k_1)$$

Condition vi: Efficiency comparison of $\widehat{M}^*_{srs}(\alpha)$ with the sample median estimator $\widehat{M}_{EP}$

By Eq. 20 and Eq. 6, $MSE_{opt}\left(\widehat{M}^*_{srs}(\alpha)\right) < MSE\left(\widehat{M}_{EP}\right)$ if

$$\frac{C^2_{M_Z}}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < \frac{C^2_{M_X}}{4}(1 + 4k_1)$$

Condition vii: Efficiency comparison of $\widehat{M}^*_{srs}(\alpha)$ with the sample median estimator $\widehat{M}_{AE}$

By Eq. 20 and Eq. 7, $MSE_{opt}\left(\widehat{M}^*_{srs}(\alpha)\right) < MSE\left(\widehat{M}_{AE}\right)$ if

$$C^2_{M_Y} + \frac{C^2_{M_Z}}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < C^2_{M_Y}(1 - \rho_{M_YM_X}{}^2)$$

Condition viii: Efficiency comparison of $\widehat{M}^*_{srs}(\alpha)$ with the sample median estimator $\widehat{M}^G_{PP}$

By Eq. 20 and Eq. 8, $MSE_{opt}\left(\widehat{M}^*_{srs}(\alpha)\right) < MSE\left(\widehat{M}^G_{PP}\right)$ if

$$C^2_{M_Y} + \frac{C^2_{M_Z}}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < \frac{\left[C^2_{M_Y}\left(1-\rho_{M_YM_X}{}^2\right)\left(1-\lambda C^2_{M_X}\right)-\frac{1}{4}C^4_{M_X}\right]}{1+\lambda M^2_Y\left(1-\rho_{M_YM_X}{}^2\right)}$$

Condition ix: Efficiency comparison of $\widehat{M}^*_{srs}(\alpha)$ with the sample median estimator $\widehat{M}^I_P$

By Eq. 20 and Eq. 9, $MSE_{opt}\left(\widehat{M}^*_{srs}(\alpha)\right) < MSE\left(\widehat{M}^I_P\right)$ if

$$C^2_{M_Y} + \frac{C^2_{M_Z}}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z} < C^2_{M_Y}\frac{\left[\left(1-\rho^2_{M_XM_Z}-\rho^2_{M_YM_X}-\rho^2_{M_YM_Z}+2\rho_{M_XM_Z}\rho_{M_YM_X}\rho_{M_YM_Z}\right)\right]}{\left(1-\rho_{M_XM_Z}{}^2\right)}$$

3.3 Application

Two different populations under simple random sampling are used to compute the numerical values for the bias and MSE of the existing and proposed estimators. The percent relative efficiencies of the estimators are obtained as follows:

$$\%RE = \frac{MSE(\widehat{M}_G)}{MSE(.)} \times 100$$

$MSE(\widehat{M}_G)$ is the MSE of the classical median estimator while $MSE(.)$ denotes the MSE of estimators mentioned here. The population statistics and the results of analyses are as follows:

Population 1: Let Population 1: Let y; x and z respectively be the number of fish caught by the marine recreational fisherman in the years 1995, 1994, and 1993 in the USA given by Singh (2003a)

$N = 69$; $n = 17$; $M_Y = 2068$; $M_X = 2011$; $M_Z = 2307$; $\rho_{M_X M_Y} = 0.1505$; $\rho_{M_X M_Z} = 0.1431$; $\rho_{M_Y M_Z} = 0.3166$; $f_Y(M_Y) = 0.00014$; $f_X(M_X) = 0.00014$; and $f_Z(M_Z) = 0.00013$.

Population 2: Let y as the U.S. exports to Singapore in billions of Singapore dollars, x as the money supply figures in billions of Singapore dollars, and z is the local supply in U.S.dollars given by Aczel and Sounderpandian (2004).

$N = 67$; $n = 23$; $M_Y = 4.8$; $M_X = 7.0$; $M_Z = 151$; $\rho_{M_X M_Y} = 0.6624$; $\rho_{M_X M_Z} = 0.7592$; $\rho_{M_Y M_Z} = 0.8624$; $f_Y(M_Y) = 0.0763$; $f_X(M_X) = 0.0526$; and $f_Z(M_Z) = 0.0024$;

Table 1. Results for Simple Random Sampling

Estimator	Population I			Population II		
	Absolute bias	MSE	PRE	Absolute bias	MSE	PRE
$\widehat{M}_G$	0	565443.6	100	0	1.23	100
$\widehat{M}_R$	246.3	988372.8	57.2	0.08	0.82	150
$\widehat{M}_{ER}$	87.27	627420.2	90.1	0.01	0.72	170.8
$\widehat{M}_{CR}$	373.78	3307296	17.1	0.59	9.31	13.2
$\widehat{M}_P$	42.32	1338418	42.2	0.17	4.06	30.3
$\widehat{M}_{EP}$	14.98	802442.8	70.5	0.05	2.34	52.6
$\widehat{M}_{AE}$	408.87	552636.1	102.3	0.03	0.69	178.3
$\widehat{M}_{PP}^G$	3373.88	491568.8	115.0	1.54	0.52	236.5
$\widehat{M}_P^I$	46089.73	502378.1	112.6	0.48	0.31	396.8
$\widehat{M}_{srs}(\alpha)$	144.55	552636.1	102.3	0.15	0.69	178.3
$\widehat{M}_{srs}^*(\alpha)$	207.98	520612.8	108.6	0.13	0.38	323.7

3.4 Median Estimators under Two Phase Sampling

Consider a finite population with N units $U = \{u_1, u_2, \dots, u_N\}$. The i^{th} unit of the population values for the study variable y, auxiliary variables x and z are y_i; x_i and z_i respectively. Under two-phase sampling design, a sample of size n is drawn using simple random sampling without replacement at first phase and the values on x and z are obtained on the units of the sample. In second phase, a simple random sampling without replacement criterion is used for drawing sample of size m from the first phase sample and the values on the variables y; x and z are taken on selected units. To obtain the properties of the proposed median estimator under two-phase sampling scheme, the following existing estimators are summarized as follows;

i) Singh, Joarder, and Tracy suggested a ratio estimator for median in two-phase Sampling

$$\widehat{M}_{SA} = \frac{\widehat{M}_y}{\widehat{M}_x}\widehat{M}'_x$$

$$B(\widehat{M}_{SA}) = \left(\frac{1}{m} - \frac{1}{n}\right)\frac{(1-\rho_{M_XM_Y})}{4f_Y(M_Y}$$

$$MSE(\widehat{M}_{SA}) = \frac{\{f_Y(M_Y)\}^{-2}}{4}\left[\left(\frac{1}{m} - \frac{1}{N}\right) + \left(\frac{1}{m} - \frac{1}{n}\right)\left(\frac{M_Yf_Y(M_Y)}{M_Xf_X(M_X)}\right)\left\{\left(\frac{\{M_Yf_Y(M_Y)}{\{M_Xf_X(M_X)}\right) - 2\rho_{M_XM_Y}\right\}\right] \quad (21)$$

ii) Singh, Singh, and Upadhyay (2007) studied a ratio-type estimator of median using two auxiliary variables

$$\widehat{M}_S = \widehat{M}_y\left(\frac{\widehat{M}'_x}{\widehat{M}_x}\right)^{\alpha_1}\left(\frac{M_Z}{\widehat{M}'_z}\right)^{\alpha_2}\left(\frac{M_Z}{\widehat{M}_z}\right)^{\alpha_3}$$

Where $\alpha_{1opt} = \left(\frac{\{M_Xf_X(M_X)}{\{M_Yf_Y(M_Y)}\right)\left(\frac{\rho_{M_XM_Z}\rho_{M_YM_Z} - \rho_{M_YM_X}}{\rho^2_{M_XM_Z} - 1}\right)$,

$$\alpha_{2opt} = \left(\frac{\{M_Zf_Z(M_Z)}{\{M_Yf_Y(M_Y)}\right)\rho_{M_XM_Z}\left(\frac{\rho_{M_XM_Z}\rho_{M_YM_Z} - \rho_{M_YM_X}}{\rho^2_{M_XM_Z} - 1}\right),\ \alpha_{3opt} = \left(\frac{\{M_Zf_Z(M_Z)}{\{M_Yf_Y(M_Y)}\right)\left(\frac{\rho_{M_XM_Z}\rho_{M_YM_X} - \rho_{M_YM_Z}}{\rho^2_{M_XM_Z} - 1}\right)$$

$$B(\widehat{M}_S) \cong \frac{\{f_Y(M_Y)\}^{-2}}{8M_Y\left(1-\rho^2_{M_XM_Z}\right)^2}\Big[\left(\frac{1}{m} - \frac{1}{n}\right)\left(1 - \rho^2_{M_XM_Z}\right)\Big\{\left(\rho_{M_YM_X} - \rho_{M_XM_Z}\rho_{M_YM_Z}\right)^2 - 2\rho_{M_XM_Y}\big(\rho_{M_YM_X} - \rho_{M_XM_Z}\rho_{M_YM_Z}\big) + \left(\frac{\{M_Yf_Y(M_Y)}{\{M_Xf_X(M_X)}\right)\left(\rho_{M_YM_X} - \rho_{M_XM_Z}\rho_{M_YM_Z}\right)\Big\} + \left(\frac{1}{m} - \frac{1}{N}\right)\left(\rho_{M_YM_Z} - \rho_{M_XM_Z}\rho_{M_YM_X}\right)\Big\{\left(\rho_{M_YM_Z} - \rho_{M_XM_Z}\rho_{M_YM_X}\right) + 2\rho_{M_XM_Z}\left(\rho_{M_YM_X} - \rho_{M_XM_Z}\rho_{M_YM_Z}\right) + \left(\frac{\{M_Yf_Y(M_Y)}{\{M_Xf_X(M_X)}\right)\left(1 - \rho^2_{M_XM_Z}\right)\Big\} + \left(\frac{1}{n} - \frac{1}{N}\right)\left(\rho_{M_YM_X} - \rho_{M_XM_Z}\rho_{M_YM_Z}\right)\Big\{\rho^2_{M_XM_Z}\left(\rho_{M_YM_X} - \rho_{M_XM_Z}\rho_{M_YM_Z}\right) - 2\rho_{M_YM_Z}\rho_{M_XM_Z}\left(1 - \rho^2_{M_XM_Z}\right) + \left(\frac{\{M_Yf_Y(M_Y)}{\{M_Xf_X(M_X)}\right)\rho_{M_XM_Z}\left(1 - \rho^2_{M_XM_Z}\right)\Big\}\Big]$$

$$MSE(\widehat{M}_S) \cong \frac{\{f_Y(M_Y)\}^{-2}}{4}\left[\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{n}-\frac{1}{N}\right)\rho^2_{M_XM_Z}-\left(\frac{1}{m}-\frac{1}{n}\right)\frac{\rho^2_{M_YM_X}+\rho^2_{M_YM_Z}-2\rho_{M_YM_X}\rho_{M_XM_Z}\rho_{M_YM_Z}}{\left(1-\rho^2_{M_XM_Z}\right)}\right] \quad (22)$$

iii) Jhajj, Kaur, and Jhajj (2016) defined ratio-exponential-type estimator as

$$\widehat{M}_{YH} = \widehat{M}_y\left(\frac{M_Z}{\widehat{M}'_Z}\right)^{v_1}\left(\frac{M_Z}{\widehat{M}_Z}\right)^{v_2} exp\left[\left(\frac{v_3(\widehat{M}_x-M_X)}{(M_X+\widehat{M}_x)}\right)\right]$$

where $v_{1opt} = \left(\frac{\{M_Zf_Z(M_Z)\}}{\{M_Yf_Y(M_Y)\}}\right)\rho_{M_XM_Z}\left(\frac{\rho_{M_XM_Z}\rho_{M_YM_Z}-\rho_{M_YM_X}}{\rho^2_{M_XM_Z}-1}\right)$,

$v_{2opt} = \left(\frac{\{M_Zf_Z(M_Z)\}}{\{M_Yf_Y(M_Y)\}}\right)\left(\frac{\rho_{M_XM_Z}\rho_{M_YM_X}-\rho_{M_YM_Z}}{\rho^2_{M_XM_Z}-1}\right)$, $v_{3opt} = \left(\frac{2\{M_Xf_X(M_X)\}}{\{M_Yf_Y(M_Y)\}}\right)\left(\frac{\rho_{M_XM_Z}\rho_{M_YM_Z}-\rho_{M_YM_X}}{\rho^2_{M_XM_Z}-1}\right)$

$$B(\widehat{M}_{YH}) \cong \frac{\{f_Y(M_Y)\}^{-2}}{8M_Y\left(1-\rho^2_{M_XM_Z}\right)^2}\left[\left(\frac{1}{m}-\frac{1}{n}\right)\left\{\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)^2-2\rho_{M_XM_Y}\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)\left(1-\rho^2_{M_XM_Z}\right)+\left(\frac{\{M_Yf_Y(M_Y)\}}{\{M_Xf_X(M_X)\}}\right)\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)\left(1-\rho^2_{M_XM_Z}\right)\right\}+\left(\frac{1}{m}-\frac{1}{N}\right)\left(\rho_{M_YM_Z}-\rho_{M_XM_Z}\rho_{M_YM_X}\right)\left\{\left(\rho_{M_YM_Z}-\rho_{M_XM_Z}\rho_{M_YM_X}\right)+2\rho_{M_XM_Z}\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)-2\rho_{M_YM_Z}\left(\rho_{M_YM_Z}-\rho_{M_XM_Z}\rho_{M_YM_X}\right)\left(1-\rho^2_{M_XM_Z}\right)+\left(\frac{\{M_Yf_Y(M_Y)\}}{\{M_Zf_Z(M_Z)\}}\right)\left(\rho_{M_YM_Z}-\rho_{M_XM_Z}\rho_{M_YM_X}\right)\left(1-\rho^2_{M_XM_Z}\right)\right\}+\left(\frac{1}{n}-\frac{1}{N}\right)\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)\rho_{M_XM_Z}\left\{\rho_{M_XM_Z}\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)-2\rho_{M_YM_Z}\left(1-\rho^2_{M_XM_Z}\right)\right\}+\left(\frac{\{M_Yf_Y(M_Y)\}}{\{M_Zf_Z(M_Z)\}}\right)\rho_{M_XM_Z}\left(\rho_{M_YM_X}-\rho_{M_XM_Z}\rho_{M_YM_Z}\right)\left(1-\rho^2_{M_XM_Z}\right)\right]$$

$$MSE(\widehat{M}_{YH}) \cong \frac{\{f_Y(M_Y)\}^{-2}}{4}\left[\left(\frac{1}{m}-\frac{1}{N}\right)-\left(\frac{1}{n}-\frac{1}{N}\right)\rho^2_{M_YM_Z}-\left(\frac{1}{m}-\frac{1}{n}\right)\frac{\rho^2_{M_YM_X}+\rho^2_{M_YM_Z}-2\rho_{M_YM_X}\rho_{M_XM_Z}\rho_{M_YM_Z}}{\left(1-\rho^2_{M_XM_Z}\right)}\right] \quad (23)$$

iv) Baig, Masood and Tarray (2019) suggested an improved class of difference-type estimators for population median under two-phase sampling with two auxiliary variables as;

$$\widehat{M}^I_P = \left[\widehat{M}_y + m_1\left(\widehat{M}'_x-\widehat{M}_x\right)\right]\left[m_2 exp\left(\frac{M_Z-\widehat{M}'_Z}{M_Z+\widehat{M}'_Z}\right)+(1-m_2)exp\left(\frac{\widehat{M}'_Z-M_Z}{M_Z+\widehat{M}'_Z}\right)\right]$$

where m_1 and m_2 are unknown constants.

$$B(\widehat{M}^I_P) = M_Y\frac{1}{4}\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{1}{2}-m_2\right)\rho_{M_YM_Z}C_{M_Y}C_{M_Z}$$

Where $m_{1(opt)} = \frac{M_YC_{M_Y}\rho_{M_YM_X}}{M_XC_{M_X}}$ and $m_{2(opt)} = \frac{1}{2}+\frac{M_Y\rho_{M_YM_Z}}{C_{M_Z}}$

$$MSE\left(\widehat{M}_P^I\right) = M_Y^2 \frac{C_{M_Y}^2}{4}\left[\left(\frac{1}{n}-\frac{1}{N}\right)+\left(\frac{1}{m}-\frac{1}{n}\right)\rho_{M_Y M_X}^2-\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{M_Y M_Z}^2\right] \tag{24}$$

3.5 Proposed Median Estimator under Two Phase Sampling

$$\widehat{M}_{srs}^D(\alpha) = \widehat{M}_y\left[\alpha\frac{\widehat{M}_x'}{\widehat{M}_x}+(1-\alpha)\frac{\widehat{M}_x}{\widehat{M}_x'}\right]exp\left[\frac{(M_Z-\widehat{M}_Z')}{(M_Z+\widehat{M}_Z')}\right] \tag{25}$$

$$\widehat{M}_{srs}^D(\alpha) = M_Y(1+e_0)[\alpha(1+e_1')(1+e_1)^{-1}+(1-\alpha)(1+e_1)(1+e_1')^{-1}\ ]exp\left\{\frac{-e_2'}{2}\left[1-\frac{e_2'}{2}+\frac{e_2'^2}{4}\right]\right\}$$

$$Bias\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y\left[\alpha(\lambda-\lambda_1)C_{M_X}^2+\frac{3}{8}\lambda_1 C_{M_Z}^2+(\lambda-\lambda_1)(1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_X M_Y}-\lambda_1\frac{C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}}{2}\right] \tag{26}$$

$$MSE\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y^2\left[(\lambda-\lambda_1)(4\alpha^2-4\alpha+1)C_{M_X}^2+\lambda_1\frac{C_{M_Z}^2}{4}+\lambda C_{M_Y}^2+2(\lambda-\lambda_1)(1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_X M_Y}-\lambda_1 C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}\right] \tag{27}$$

For optimum value of the MSE of $\widehat{M}_{srs}^D(\alpha)$, the value of α is given as $\alpha = \frac{k_1+1}{2}$, then (27) becomes

$$MSE_{opt}\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y^2\left[\lambda C_{M_Y}^2-(\lambda-\lambda_1)C_{M_Y}^2\rho_{M_X M_Y}^2+\lambda_1\frac{C_{M_Z}^2}{4}-\lambda_1 C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}\right] \tag{28}$$

In addition, the optimum bias becomes

$$Bias_{opt}\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y\left[(\lambda-\lambda_1)\frac{C_{M_X}^2}{2}+(\lambda-\lambda_1)\frac{C_{M_X}C_{M_Y}\rho_{M_X M_Y}}{2}+\frac{3}{8}\lambda_1 C_{M_Z}^2-(\lambda-\lambda_1)C_{M_Y}^2\rho_{M_X M_Z}^2-\lambda_1\frac{C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}}{2}\right] \tag{29}$$

3.6 Efficiency Comparison of Estimators under Two Phase Sampling

Condition I: Efficiency comparison of $\widehat{M}_{srs}^D(\alpha)$ with the sample median estimator $\widehat{M}_{SA}$

By Eq. 28 and Eq. 21, $MSE_{opt}\left(\widehat{M}_{srs}^D(\alpha)\right) < MSE\left(\widehat{M}_{SA}\right)$ if

$$\left(\frac{1}{m}-\frac{1}{n}\right)\left[C_{M_Y}^2\left(1+\rho_{M_X M_Y}^2\right)-C_{M_X}^2\right]-\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{C_{M_Z}^2}{4}-C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}\right) > 2\left(\frac{1}{m}-\frac{1}{n}\right)C_{M_X}C_{M_Y}\rho_{M_X M_Y}$$

Condition II: Efficiency comparison of $\widehat{M}_{srs}^{D}(\alpha)$ with the sample median estimator $\widehat{M}_S$

By Eq. 28 and Eq. 22, $MSE_{opt}\left(\widehat{M}_{srs}^{D}(\alpha)\right) < MSE\left(\widehat{M}_S\right)$ if

$$\left(\frac{1}{m}-\frac{1}{n}\right)\left[C_{M_Y}^2 \frac{\left(1-\rho_{M_XM_Z}^2+2\rho_{M_XM_Y}^2-\rho_{M_XM_Y}^2\rho_{M_XM_Z}^2+\rho_{M_YM_Z}^2-2\rho_{XM_Y}\rho_{M_XM_Z}\rho_{M_YM_Z}\right)}{1-\rho_{M_XM_Z}^2}\right]-\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{C_{M_Z}^2}{4}-C_{M_Y}C_{M_Z}\rho_{M_YM_Z}\right) > \left(\frac{1}{n}-\frac{1}{N}\right)C_{M_Y}^2\rho_{M_XM_Z}^2$$

N/B: The same condition holds for Eq. 23, since $MSE\left(\widehat{M}_S\right) = MSE\left(\widehat{M}_{YH}\right)$

Condition III: Efficiency comparison of $\widehat{M}_{srs}^{D}(\alpha)$ with the sample median estimator $\widehat{M}_P^I$

By Eq. 28 and Eq. 24, $MSE_{opt}\left(\widehat{M}_{srs}^{D}(\alpha)\right) < MSE\left(\widehat{M}_P^I\right)$ if

$$\left(\frac{1}{m}-\frac{1}{N}\right)\left(\frac{C_{M_Z}^2}{4}-C_{M_Y}C_{M_Z}\rho_{M_YM_Z}\right)-2\left(\frac{1}{m}-\frac{1}{n}\right)C_{M_Y}^2\rho_{M_YM_X}^2+\left(\frac{1}{m}-\frac{1}{N}\right)C_{M_Y}^2\rho_{M_YM_Z}^2 < 0$$

3.7 Numerical study under two-phase sampling

Here, three different populations will be in use to validate the theoretical claims of both the existing and the proposed estimators. The population statistics and the results of analyses are as follows:

Population 3: Let Population 1: Let y; x and z respectively be the number of fish caught by the marine recreational fisherman in the years 1995, 1994 and 1993 in the USA given by Singh (2003a)
$N = 69$; $n' = 24$ $n = 17$; $M_Y = 2068$; $M_X = 2011$; $M_z = 2307$; $\rho_{M_XM_Y} = 0.1505$; $\rho_{M_XM_Z} = 0.1431$; $\rho_{M_YM_Z} = 0.3166$; $f_Y(M_Y) = 0.00014$; $f_X(M_X) = 0.00014$; and $f_Z(M_Z) = 0.00013$.

Population 4: Let y as the U.S. exports to Singapore in billions of Singapore dollars, x as the money supply figures in billions of Singapore dollars, and z is the local supply in U.S. dollars given by Aczel and Sounderpandian (2004).
$N = 67$; $n' = 23$; $n = 15$; $M_Y = 4.8$; $M_X = 7.0$; $M_Z = 151$; $\rho_{M_XM_Y} = 0.6624$; $\rho_{M_XM_Z} = 0.7592$; $\rho_{M_YM_Z} = 0.8624$; $f_Y(M_Y) = 0.0763$; $f_X(M_X) = 0.0526$; and $f_Z(M_Z) = 0.0024$;

Population 5: Let y be the district-wise tomato production (tones) in 2003, x as a district-wise tomato production (tones) in 2002, and z as a district-wise tomato production (tones) in 2001 given by MFA (2004).

$N = 97$; $n' = 46$; $n = 33$; $M_Y = 1242$; $M_X = 1233$; $M_Z = 1207$; $\rho_{M_X M_Y} = 0.2096$; $\rho_{M_X M_Z} = 0.15$; $\rho_{M_Y M_Z} = 0.123$; $f_Y(M_Y) = 0.00021$; $f_X(M_X) = 0.0002$; and $f_Z(M_Z) = 0.0002$;

From the numerical study of the three real-life data sets, the following remarks are deduced;

Table 2. Results for Two Phase Sampling

Estimator	Population III			Population IV			Population V		
	AB	MSE	PRE	AB	MSE	PRE	AB	MSE	PRE
$\widehat{M}_G$	0	346606	100	0	1.23	100	0	64795.08	100
$\widehat{M}_{SA}$	26.05	729125.3	47.54	0.03	1.89	65.08	8.06	146126.9	44.34
$\widehat{M}_S$	58.72	506293.8	68.46	0.35	0.57	215.79	8.49	109813.8	59.00
$\widehat{M}_{YH}$	259.20	506293.8	68.46	0.09	0.57	215.79	286.51	109813.8	59.00
$\widehat{M}_P^I$	25.73	523809	66.17	0.35	0.13	946.13	1.41	109495.7	58.18
$\widehat{M}_{srs}^D(\alpha)$	8.20	310481.2	111.635	0.08	0.29	424.14	8.57	84943.4	76.28

N/B: AB=Absolute bias

4.0 Discussion

The result in Table 1 shows that the proposed estimator $\widehat{M}_{srs}^*(\alpha)$ has outperformed other existing estimators considered in this study in terms of minimum bias and gains in efficiency concerning the given set of data. However, the existing estimator, $\widehat{M}_P^I$ had slightly gain in efficiency but heavily biased in both population I and II, respectively. Again, as shown in Table 2, the proposed estimator $\widehat{M}_{srs}^D(\alpha)$ has an overwhelming performance in terms of greater gains in efficiency and minimum bias than the existing estimators considered in this study. This superiority of the proposed estimator is because of the endearing properties of the separate ratio product–type estimator. However, $\widehat{M}_P^I$, noticeably from populations IV, had a slight gains in efficiency over the proposed estimator but, was highly biased and not suitable for the estimation of population median, given unbiasedness as an objective of the study. It becomes imperative to seek for an estimator with minimum biasedness and greater efficiency that will enhance the estimation of the median of a finite population of which the proposed estimator has bridged the gap.

5.0 Conclusion

This study was towards formulating an improved exponential-type estimator for population median using both simple random and two-phase sampling with two auxiliary variables. The bias and mean square error were obtained. From the efficiency comparisons and numerical illustrations, it is evident to say that among the existing and proposed estimators, the proposed estimators under simple random sampling and two-phase sampling exhibit superior performance in negligible bias and gains in efficiency. It suffices to conclude that the proposed estimators perform better than the existing estimators do, as per this study, in estimating population median when the population median of the auxiliary variables are positively correlated with the study variable. However, though the suggested estimator performed poorly behind $\widehat{M}_P^I$ with less efficiency in population IV, probably because of the data type, it shows to be asymptotically unbiased compared to other estimators of its class considered in this study. The idea of using a separate ratio-product exponential-type estimator, in this case, has paid off in improving the efficiency of the median estimator under simple random and two-phase sampling with two auxiliary variables. In addition, the proposed estimator will be suitable and highly recommended when the variable considered is from a distribution that is highly skewed.

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