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Few More Identities of Rogers-Ramanujan Type

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ABSTRACT: We derive some Identities of Rogers-Ramanujan Type modulo 3s, 10s and 15s where s is any finite positive integer. Particular cases for such identities modulo 20, 30 and 45 also have been derived.

Key Words: Rogers-Ramanujan Identity, Basic Hypergeometric Series, Jacobi's Triple Product Identity, Bailey's Lemma etc.

1. **Introduction:** The following two identities, namely for $|q|<1$,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\, n\not\equiv 0,\pm 2\,(mod\ 5\,)$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\, n\not\equiv 0,\pm 1(mod\ 5\,)$$

where $(q;q)_n=(1-q)(1-q^2)...(1-q^n)$,

are called the Rogers-Ramanujan Identities and they have motivated extensive research work over the past hundred years. In the recent decades, several eminent mathematicians like W. N. Bailey, G.N. Watson, G.E. Andrews, A. Verma and V.K Jain have derived many identities of the Rogers-Ramanujan Type. In this paper, we have derived few more identities of the Rogers-Ramanujan Type.

For |q|<1, the q −shifted factorial is defined by

$$(a;q)_0=1$$

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\ \text{for } n\geq 1$$

$$\text{and}\quad (a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^k).$$

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorials is defined by

$$(a_1, a_2, \ldots\ldots a_m; q)_n = (a_1;q)_n(a_2;q)_n \ldots.(a_m;q)_n$$

$$(a_1, a_2, \ldots\ldots a_m; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \ldots.(a_m;q)_\infty$$

The Basic Hyper geometric Series is

$$ {}_{p+1}\phi_{p+r}\left(\begin{matrix} a_1, a_2, \ldots\ldots, a_{p+1}; q; x \\ b_1, b_2, \ldots\ldots, b_{p+r} \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_{p+1};q)_n x^n (-1)^{nr} q^{\frac{n(n-1)r}{2}}}{(q;q)_n (b_1:q)_n (b_2;q)_n \ldots\ldots (b_{p+r};q)_n} $$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r = 0 it converges only when $|x|<1$.

The q-analogue of Saalchutz Theorem is

$$ {}_3\phi_2\left(\begin{matrix} e, f, q^{-n}; q \\ \frac{aq}{c}, \frac{cefq^{-n}}{a} \end{matrix}\right) = \frac{\left(\frac{aq}{ec}\right)_n \left(\frac{aq}{cf}\right)_n}{\left(\frac{aq}{c}\right)_n \left(\frac{aq}{cef}\right)_n} \qquad (1.1)$$

We require the following Jacobi's Triple Product Identity (See [3], 2.2.10 and 2.2.11)

$(zq^{1/2}, z^{-1}q^{1/2}, q; q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\frac{n^2}{2}}$, and its corollary is given by

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2} - in} = \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)-in}.(1 - q^{(2n+1)i})$$

$$= \prod_{n=0}^{\infty}(1 - q^{(2k+1)(n+1)})(1 - q^{(2k+1)n+i})(1 - q^{(2k+1)(n+1)-i}) \qquad (1.2)$$

The following Lemma is due to W. N. Bailey: (See [1], (2.1.5))

If p is a non-negative integer, then

$$(aq;q)_\infty \sum_{n=0}^{\infty} a^n.q^{n^2-pn}.\beta_n = \sum_{j=0}^{p} \frac{(q^{-p};q)_j (-a)^j q^{j(j+1)/2}}{(q;q)_j}.\sum_{n=0}^{\infty} a^n.q^{n^2-pn+2nj}.\alpha_n \qquad (1.3)$$

where $\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q;q)_{n-k}(aq;q)_{n+k}}$

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2. We begin by introducing the following transformations:

$$
{}_{12}\phi_{11}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, x, \omega x, \omega^2 x, y, \omega y, \omega^2 y, q^{-n}, \omega q^{-n}, \omega^2 q^{-n}; q; \dfrac{a^4 q^{4+3n}}{x^3 y^3} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{x}, \dfrac{aq\omega^2}{x}, \dfrac{aq\omega}{x}, \dfrac{aq}{y}, \dfrac{aq\omega^2}{y}, \dfrac{aq\omega}{y}, aq^{1+n}, \omega^2 aq^{1+n}, \omega aq^{1+n} \end{matrix}\right)
$$

$$
= \frac{(a^3q^3;q^3)_n \left(\dfrac{a^3q^3}{x^3y^3};q^3\right)_n}{\left(\dfrac{a^3q^3}{x^3};q^3\right)_n \left(\dfrac{a^3q^3}{y^3};q^3\right)_n} \cdot {}_6\phi_5\left(\begin{matrix} aq, aq^2, aq^3, x^3, y^3, q^{-3n}; q^3; q^3 \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^3, -a^{3/2}q^3, \dfrac{x^3y^3}{a^3}q^{-3n} \end{matrix}\right) \qquad \textbf{(2.1)}
$$

$$
{}_{12}\phi_{11}\left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, q^{-n}, q^{-n+1}, q^{-n+2}; q^3; \dfrac{a^4 q^{3+3n}}{x^3 y^3} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq^3}{x}, \dfrac{aq^2}{x}, \dfrac{aq}{x}, \dfrac{aq^3}{y}, \dfrac{aq^2}{y}, \dfrac{aq}{y}, aq^{3+n}, aq^{2+n}, aq^{1+n} \end{matrix}\right)
$$

$$
= \frac{(aq;q)_n \left(\dfrac{aq}{xy};q\right)_n}{\left(\dfrac{aq}{x};q\right)_n \left(\dfrac{aq}{y};q\right)_n} \cdot {}_6\phi_5\left(\begin{matrix} a^{\frac{1}{3}}, \omega a^{\frac{1}{3}}, \omega^2 a^{\frac{1}{3}}, x, y, q^{-n}; q; q \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, \dfrac{xy}{a}q^{-n} \end{matrix}\right) \qquad (2.2)
$$

Proof of (2.1) and (2.2): (See [1], (1.5) and (1.6) respectively)

The multiple series generalization of (2.2) can be given in the form: (See [1], (4.5)).

$$
{}_{2p+4}\phi_{2p+3}\left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, (c_{p-4}), (d_{p-4}), q^{-n}, q^{-n+1}, q^{-n+2}; q^3; \dfrac{a^p q^{3p-9+3n}}{x^3 y^3 c_1 d_1 \ldots c_{p-4} d_{p-4}} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq^3}{x}, \dfrac{aq^2}{x}, \dfrac{aq}{x}, \dfrac{aq^3}{y}, \dfrac{aq^2}{y}, \dfrac{aq}{y}, \dfrac{aq^3}{(c_{p-4})}, \dfrac{aq^3}{(d_{p-4})}, aq^{3+n}, aq^{2+n}, aq^{1+n} \end{matrix}\right)
$$

$$
= \frac{(aq;q)_n \left(\dfrac{aq}{xy};q\right)_n}{\left(\dfrac{aq}{x};q\right)_n \left(\dfrac{aq}{y};q\right)_n} \sum_{r_1, r_2, \ldots r_{p-4}} \prod_{j=1}^{p-4} \left\{ \frac{\left(\dfrac{aq^3}{c_j d_j};q^3\right)_{r_j} (c_j;q^3)_{M_{j-1}} (d_j;q^3)_{M_{j-1}}}{(q^3;q^3)_{r_j} \left(\dfrac{aq^3}{c_j};q^3\right)_{M_j} \left(\dfrac{aq^3}{d_j};q^3\right)_{M_j}} \right\}.
$$

$$.\frac{(x;q)_{3M_{p-4}}(y;q)_{3M_{p-4}}(q^{-n};q)_{3M_{p-4}}(aq^3;q^3)_{2M_{p-4}}q^{3M_{p-4}(M_{p-4}+1)}}{(aq;q)_{6M_{p-4}}\left(\frac{xy}{a}q^{-n};q\right)_{3M_{p-4}}}.$$

$$(a)^{r_{p-4}}\left(\frac{a^2q^3}{c_{p-4}d_{p-4}}\right)^{r_{p-5}}\dots\left(\frac{a^{p-4}q^{3p-15}}{c_{p-4}.d_{p-4}\dots c_2d_2}q^2\right)^{r_1}.$$

$${}_6\phi_5\left(\begin{matrix}a^{\frac{1}{3}}q^{2M_{p-4}},\omega a^{\frac{1}{3}}q^{2M_{p-4}},\omega^2a^{\frac{1}{3}}q^{2M_{p-4}},xq^{3M_{p-4}},yq^{3M_{p-4}},q^{-n+3M_{p-4}};q;q\\ a^{\frac{1}{2}}q^{3M_{p-4}},-a^{\frac{1}{2}}q^{3M_{p-4}},a^{\frac{1}{2}}q^{\frac{1}{2}+3M_{p-4}},-a^{\frac{1}{2}}q^{\frac{1}{2}+3M_{p-4}},\frac{xy}{a}q^{-n+3M_{p-4}}\end{matrix}\right) \tag{2.3}$$

for $p \geq 4$ (ω is the imaginary cube root of unity), and $M_i = r_1 + r_2 + \cdots + r_i$, $M_{-1} = M_0 = 0$

Main Results:

43. *Rogers-Ramanujan Type Identities Modulo 15s: (where s is any finite positive integer)*

Replacing $\boldsymbol{q}$ by $\boldsymbol{q^s}$ in (2.1), it can be written as,

$$\sum_{k=0}^{n}\frac{(a;q^s)_k(a^2q^{2s};q^{2s})_k(x^3;q^{3s})_k(y^3;q^{3s})_k(q^{-3ns};q^{3s})_k\left(\frac{a^4q^{4s+3ns}}{x^3y^3}\right)^k}{(q^s;q^s)_k(a^2;q^{2s})_k\left(\frac{a^3q^{3s}}{x^3};q^{3s}\right)_k\left(\frac{a^3q^{3s}}{y^3};q^{3s}\right)_k(a^3q^{3s+3ns};q^{3s})_k}$$

$$=\frac{\left(a^3q^{3s};q^{3s}\right)_n\left(\frac{a^3q^{3s}}{x^3y^3};q^{3s}\right)_n}{\left(\frac{a^3q^{3s}}{x^3};q^{3s}\right)_n\left(\frac{a^3q^{3s}}{y^3};q^{3s}\right)_n}\sum_{k=0}^{n}\frac{(aq^s;q^{3s})_k(aq^{2s};q^{3s})_k(aq^{3s};q^{3s})_k(x^3;q^{3s})_k(y^3;q^{3s})_k(q^{-3ns};q^{3s})_k.q^{3k}}{(q^{3s};q^{3s})_k(a^3q^{3s};q^{6s})_k(a^3q^{6s};q^{6s})_k\left(\frac{x^3y^3q^{-3ns}}{a^3};q^{3s}\right)_k} \tag{3.1}$$

Taking $x, y \to \infty$ (3.1), we find after some simplification, the following equation,

$$\sum_{k=0}^{n}\frac{(-1)^k(aq^s;q^s)_{k-1}(1-a^2q^{2ks})a^{4k}q^{\frac{9k^2s-ks}{2}}}{(q^s;q^s)_k(1+a)(a^3q^{3s};q^{3s})_{n+k}(q^{3s};q^{3s})_{n-k}} = \sum_{k=0}^{n}\frac{a^{3k}(aq^s;q^s)_{3k}\,q^{3k^2s}}{(q^{3s};q^{3s})_k(a^3q^{3s};q^{3s})_{2k}(q^{3s};q^{3s})_{n-k}} \tag{3.2}$$

The left hand side of Bailey's Lemma (1.3) for $a = a^3$, $q = q^{3s}$, $\alpha_0 = 1$, $\alpha_{k+1} = 0$ and,

$$\alpha_k = \frac{(-1)^k(aq^s;q^s)_{k-1}(1-a^2q^{2ks})a^{4k}q^{\frac{9k^2s-ks}{2}}}{(q^s;q^s)_k(1+a)},$$ gives,

$$(a^3q^{3s};q^{3s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a^{3n+6k}\,(aq^s;q^s)_{3k}\,q^{3(n+k)^2s-3p(n+k)s+3k^2s}}{(q^{3s};q^{3s})_k(a^3q^{3s};q^{3s})_{2k}(q^{3s};q^{3s})_n}\quad \text{, (upon using (3.2)}\qquad (3.3)$$

The corresponding Right hand side of Bailey's Lemma (1.3) for the same, yields

$$\sum_{j=0}^{p}\frac{(q^{-3ps};q^{3s})_j\,(-1)^j a^{3j} q^{\frac{3j(j+1)s}{2}}}{(q^{3s};q^{3s})_j}.\sum_{n=0}^{\infty}\frac{(-1)^n a^{7n}(aq^s;q^s)_{n-1}(1-a^2q^{2ns})q^{\frac{15n^2s-ns-6pns+12njs}{2}}}{(q^s;q^s)_n(1+a)}\qquad (3.4)$$

Equating (3.3) and (3.4), we obtain,

$$(a^3q^{3s};q^{3s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a^{3n+6k}\,(aq^s;q^s)_{3k}\,q^{3(n+k)^2s-3p(n+k)s+3k^2s}}{(q^{3s};q^{3s})_k(a^3q^{3s};q^{3s})_{2k}(q^{3s};q^{3s})_n}$$

$$=\sum_{j=0}^{p}\frac{(q^{-3ps};q^{3s})_j\,(-1)^j a^{3j} q^{\frac{3j(j+1)s}{2}}}{(q^{3s};q^{3s})_j}.\sum_{n=0}^{\infty}\frac{(-1)^n a^{7n}(aq^s;q^s)_{n-1}(1-a^2q^{2ns})q^{\frac{15n^2s-ns-6pns+12njs}{2}}}{(q^s;q^s)_n(1+a)}\qquad (3.5)$$

Setting $a = 1$ and placing $p = 0, 1$ successively in (3.5), we get the following Identities upon using (1.2):

$$\frac{(q^{3s};q^{3s})_\infty}{(q^s;q^s)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^s;q^s)_{3k}\,q^{3(n+k)^2s+3k^2s}}{(q^{3s};q^{3s})_k(q^{3s};q^{3s})_{2k}(q^{3s};q^{3s})_n}$$

$$=\frac{1}{2(q^s;q^s)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(q^s;q^s)_{n-1}(1-q^{2ns})q^{\frac{15n^2s-ns}{2}}}{(q^s;q^s)_n}$$

$$=\frac{1}{2(q^s;q^s)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{15n^2s-ns}{2}}$$

$$=\frac{1}{2}\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ \text{where } n\not\equiv 0,\pm 8s \pmod{15s}\qquad (3.6)$$

$$2\,\frac{(q^{3s};q^{3s})_\infty}{(q^s;q^s)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^s;q^s)_{3k}\,q^{3(n+k)^2s-3(n+k)s+3k^2s}}{(q^{3s};q^{3s})_k(q^{3s};q^{3s})_{2k}(q^{3s};q^{3s})_n}$$

$$=\frac{1}{(q^s;q^s)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{15n^2s+7ns}{2}}+\frac{1}{(q^s;q^s)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{15n^2s+5ns}{2}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}\qquad + \qquad \prod_{n=1}^{\infty}\frac{1}{1-q^n}\qquad (3.7)$$

where $n\not\equiv 0,4s \pmod{15s}$, $\quad n\not\equiv 0,5s \pmod{15s}$

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4. Rogers-Ramanujan Type Identities Modulo 3s: (where s is any finite positive integer)

Taking $x, y \to 0$ in (4.1) and then replacing $\boldsymbol{q}$ by $\boldsymbol{q^{-s}}$, we find that,

$$\sum_{k=0}^{n} \frac{(-1)^k a^{-2k}(aq^{-3s};q^{-3s})_{k-1}(1-aq^{-6ks})q^{\frac{(9k^2s+3ks)}{2}}}{(q^{-3s};q^{-3s})_k(aq^{-s};q^{-s})_{n+3k}(q^{-s};q^{-s})_{n-3k}}$$

$$= \frac{(-1)^n a^{-n} q^{(n^2s+ns)/2}}{(q^{-s};q^{-s})_n} . \sum_{k=0}^{n} \frac{(-1)^k (aq^{-3s};q^{-3s})_{k-1}\, q^{\frac{-k^2s-3ks+2nks}{2}}}{(q^{-s};q^{-s})_k(aq^{-2s};q^{-2s})_{k-1}(aq^{-s};q^{-2s})_{2k}(q^{-s};q^{-s})_{n-k}} \quad (4.1)$$

The Bailey's Lemma (1.3) for $q = q^{-2s}$, $\alpha_0 = 1$, $\alpha_{3k+1} = 0$ and,

$$\alpha_{3k} = \frac{(-1)^k a^{-2k}(aq^{-6s};q^{-6s})_{k-1}(1-aq^{-12ks})q^{(9k^2s+3ks)}}{(q^{-6s};q^{-6s})_k},$$ gives (upon using (4.1)),

$$(aq^{-s};q^{-s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+2k}(aq^{-3s};q^{-3s})_{k-1} . q^{\frac{-(n+k)^2s+(n+k)s+2p(n+k)s+k^2s-3ks+2nks}{2}}}{(q^{-s};q^{-s})_k(aq^{-2s};q^{-2s})_{k-1}(aq^{-s};q^{-2s})_{2k}(q^{-s};q^{-s})_n}$$

$$= \sum_{j=0}^{p} \frac{(q^{ps};q^{-s})_j (-a)^j q^{-j(j+1)s/2}}{(q^{-s};q^{-s})_j} . \sum_{n=0}^{\infty} . \frac{(-1)^n a^n (aq^{-3s};q^{-3s})_{n-1}\,(1-aq^{-6ns})q^{\frac{(3n^2s+3ns+6pns-12njs)}{2}}}{(q^{-3s};q^{-3s})_n} \quad (4.2)$$

Setting $a = 1$ and then placing $p = 0, 1$ successively in (4.2), we get the following Identities upon using (1.2):

$$\frac{(q^{-s};q^{-s})_\infty}{(q^s;q^s)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+2k}(q^{-3s};q^{-3s})_{k-1}\, q^{\frac{-(n+k)^2s+(n+k)s+k^2s-3ks+2nks}{2}}}{(q^{-s};q^{-s})_{n+k}(q^{-s};q^{-s})_k(q^{-2s};q^{-2s})_{k-1}(q^{-s};q^{-2s})_{2k}(q^{-s};q^{-s})_n}$$

$$= \frac{1}{(q^s;q^s)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{(3n^2s+3ns}{2}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0 \pmod{3s} \quad (4.3)$$

$$\frac{(q^{-s};q^{-s})_\infty}{(q^s;q^s)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+2k}(q^{-3s};q^{-3s})_{k-1}\, q^{\frac{-(n+k)^2s+3(n+k)s+k^2s-3ks+2nks}{2}}}{(q^{-s};q^{-s})_k(q^{-2s};q^{-2s})_{k-1}(q^{-s};q^{-2s})_{2k}(q^{-s};q^{-s})_n}$$

$$= \frac{1}{(q^s;q^s)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{3n^2s+3ns}{2}} + \frac{1}{(q^s;q^s)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{3n^2s+9ns}{2}}$$

$$= 2. \prod_{n=1}^{\infty} \frac{1}{1-q^n} \text{ , where } n \not\equiv 0 \pmod{3s} \quad (4.3)$$

5. Rogers-Ramanujan Type Identities Modulo 10s: (where s is any finite positive integer)

Setting $p = 5, c_1 = z, d_1 = zq$ in the transformation (2.3), we get,

$$ {}_{14}\phi_{13}\left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, z, zq, q^{-n}, q^{-n+1}, q^{-n+2}; q^3; \dfrac{a^5q^{6+3n}}{x^3y^3z^2q} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq^3}{x}, \dfrac{aq^2}{x}, \dfrac{aq}{x}, \dfrac{aq^3}{y}, \dfrac{aq^2}{y}, \dfrac{aq}{y}, \dfrac{aq^3}{z}, \dfrac{aq^2}{z}, aq^{3+n}, aq^{2+n}, aq^{1+n} \end{matrix}\right) $$

$$ = \frac{(aq;q)_n\left(\frac{aq}{xy};q\right)_n}{\left(\frac{aq}{x};q\right)_n\left(\frac{aq}{y};q\right)_n} \sum_{r=0}^{\infty} \frac{(\frac{aq^2}{z^2};q^3)_r (x;q)_{3r} (y;q)_{3r} (q^{-n};q)_{3r} (aq^3;q^3)_{2r} q^{3r(r+1)} a^r}{(q^3;q^3)_r (\frac{aq^3}{z};q^3)_r (\frac{aq^2}{z};q^3)_r (aq;q)_{6r} (\frac{xyq^{-n}}{a};q)_{3r}} \times. $$

$$ {}_6\phi_5\left(\begin{matrix} a^{\frac{1}{3}}q^{2r}, \omega a^{\frac{1}{3}}q^{2r}, \omega^2 a^{\frac{1}{3}}q^{2r}, xq^{3r}, yq^{3r}, q^{-n+3r}; q; q \\ a^{\frac{1}{2}}q^{3r}, -a^{\frac{1}{2}}q^{3r}, a^{\frac{1}{2}}q^{\frac{1}{2}+3r}, -a^{\frac{1}{2}}q^{\frac{1}{2}+3r}, \dfrac{xy}{a}q^{-n+3r} \end{matrix}\right) \tag{5.1} $$

Letting $n, x, y, z \to \infty$ in (5.1) and then replacing q by $q^{2s/3}$, we get,

$$ (aq^{\frac{2s}{3}}; q^{\frac{2s}{3}})_\infty \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(aq^{2s};q^{2s})_{2r+k-1} a^r . q^{8r^2s+\frac{2}{3}k^2s+4rks}}{(aq^{\frac{2s}{3}};q^{\frac{2s}{3}})_{2k+6r} (q^{\frac{2s}{3}};q^{\frac{2s}{3}})_k (q^{2s};q^{2s})_r} $$

$$ = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k} (aq^{2s};q^{2s})_{k-1} (1-aq^{4ks})\, q^{5k^2s-ks}}{(q^{2s};q^{2s})_k} \tag{5.2} $$

Putting $a = 1, q^2$ successively in (5.2), we get the following two identities upon using (1.2):

$$ \frac{(q^{\frac{2s}{3}};q^{\frac{2s}{3}})_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q^{2s};q^{2s})_{2r+k-1} . q^{8r^2s+\frac{2}{3}k^2s+4rks}}{(q^{\frac{2s}{3}};q^{\frac{2s}{3}})_{2k+6r} (q^{\frac{2s}{3}};q^{\frac{2s}{3}})_k (q^{2s};q^{2s})_r} = \frac{1}{(q;q)_\infty} . \sum_{k=0}^{\infty} \frac{(-1)^k (q^{2s};q^{2s})_{k-1} (1-q^{4ks})\, q^{5k^2s-ks}}{(q^{2s};q^{2s})_k} $$

$$ = \frac{1}{(q;q)_\infty} . \sum_{k=-\infty}^{\infty} (-1)^k\, q^{5k^2s+ks} $$

$$ = \prod_{k=1}^{\infty} \frac{1}{1-q^k}, \text{ where } k \not\equiv 0, \pm 4s \pmod{10s} \tag{5.3} $$

$$ \frac{(q^{2s};q^{\frac{2s}{3}})_\infty}{(q;q)_\infty} \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q^{4s};q^{2s})_{2r+k-1} . q^{8r^2s+\frac{2}{3}k^2s+4rks+2rs}}{(q^{\frac{8s}{3}};q^{\frac{2s}{3}})_{2k+6r} (q^{\frac{2s}{3}};q^{\frac{2s}{3}})_k (q^{2s};q^{2s})_r} $$

$$=\frac{1}{(q;q)_\infty}.\sum_{k=-\infty}^{\infty}(-1)^k\, q^{5k^2s+3ks}$$

$$=\prod_{k=1}^{\infty}\frac{1}{1-q^k},\text{ where } k\not\equiv 0,\ \pm 2s \text{ (mod 10s)} \qquad (5.4)$$

6. Particular cases:

a. *Identities Modulo 20:*

Letting s = 2 successively in (5.3) and (5.4), we get

$$\frac{(q^{\frac{4}{3}};q^{\frac{4}{3}})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^4;q^4)_{2r+k-1}.q^{16r^2+\frac{4}{3}k^2+8rk}}{(q^{\frac{4}{3}};q^{\frac{4}{3}})_{2k+6r}\,(q^{\frac{4}{3}};q^{\frac{4}{3}})_k\,(q^4;q^4)_r}=\prod_{k=1}^{\infty}\frac{1}{1-q^k},\text{ where } k\not\equiv 0,\ \pm 8 \text{ (mod 20)} \qquad (6.1)$$

$$\frac{(q^4;q^{\frac{4}{3}})_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^8;q^4)_{2r+k-1}.q^{16r^2+\frac{4}{3}k^2+8rk+4r}}{(q^{\frac{16}{3}};q^{\frac{4}{3}})_{2k+6r}\,(q^{\frac{4}{3}};q^{\frac{4}{3}})_k\,(q^4;q^4)_r}=\prod_{k=1}^{\infty}\frac{1}{1-q^k},\text{ where } k\not\equiv 0,\ \pm 4 \text{ (mod 20)} \qquad (6.2)$$

b. *Identities Modulo 30:*

Letting s = 2 successively in (3.6) and (3.7), we get

$$\frac{2.(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^2;q^2)_{3k}\,q^{6n^2+12nk+12k^2}}{(q^6;q^6)_k(q^6;q^6)_{2k}(q^6;q^6)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\text{ where } n\not\equiv 0,\pm 16 \text{ (mod 30)} \qquad (6.3)$$

$$\frac{2.(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^2;q^2)_{3k}\,q^{6n^2+12nk+12k^2-6n-6k}}{(q^6;q^6)_k(q^6;q^6)_{2k}(q^6;q^6)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}\quad+\quad\prod_{n=1}^{\infty}\frac{1}{1-q^n} \qquad (6.4)$$

where $n\not\equiv 0,8$ (mod 30) and $n\not\equiv 0,10$ (mod 30) respectively.

Letting s = 3 successively in (5.3) and (5.4), we get

$$(-q;q)_\infty\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^6;q^6)_{2r+k-1}.q^{24r^2+2k^2+12rk}}{(q^2;q^2)_{2k+6r}\,(q^2;q^2)_k\,(q^6;q^6)_r}=\prod_{k=1}^{\infty}\frac{1}{1-q^k},\text{ where } k\not\equiv 0,\ \pm 12 \text{ (mod 30)} \qquad (6.5)$$

$$(-q;q)_\infty\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^6;q^6)_{2r+k}\,.q^{24+2k^2+12rk+6r}}{(q^2;q^2)_{2k+6r+3}\,(q^2;q^2)_k\,(q^4;q^4)_r}=\prod_{k=1}^{\infty}\frac{1}{1-q^k},\text{ where } k\not\equiv 0,\ \pm 6 \text{ (mod 30)} \qquad (6.6)$$

c. *Identities Modulo 45:*

Letting s = 3 successively in (3.6) and (3.7), we get

$$\frac{2.(q^9;q^9)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^3;q^3)_{3k}\,q^{9n^2+18nk+9k^2}}{(q^9;q^9)_k(q^9;q^9)_{2k}(q^9;q^9)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\text{ where } n\not\equiv 0,\pm 24 \text{ (mod 45)} \qquad (6.7)$$

$$\frac{2.(q^9;q^9)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^3;q^3)_{3k}\,q^{9n^2+18nk+18k^2-9n-9k}}{(q^9;q^9)_k(q^9;q^9)_{2k}(q^9;q^9)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n} \quad + \quad \prod_{n=1}^{\infty}\frac{1}{1-q^n} \qquad (6.8)$$

where $n \not\equiv 0,12 \pmod{45}$ and $n \not\equiv 0,15 \pmod{45}$ respectively.

7. Conclusion: The transformation (2.3) which is the series generalization of (2.2) can also be used for searching further identities of the Rogers-Ramanujan type in this line for some higher values of p. Researchers can also go through the generalization of the transformation (2.1) which is available in the work of Verma and V.K. Jain [1].

Reference:

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