Review on Numerical methods for differential equation of fractional order and computer based algorithms

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Abstract

Because finding the exact solution of such random fractional differential equations is not straightforward in comparison to the usual integer ordinal differential equations and in addition to the famous numerical methods we use, we will review the numerical methods used to solve fractional differential equations in this article. Check out the algorithms used by computer languages for solving fractional differential equations. As there are many numerical codes available for solving integer order differential equations in a variety of computing systems (e.g., Maple, Mathematica, MATLAB, Python, etc.), but fewer for solving fractional ordinal differential equations, we will compare and review some of these here.

Keywords:differential equations, fractional integral equation; Fractional derivatives and Integrals,Numerical methods, algorithms.

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1 Introduction

Fractional calculus is as old as calculus, which was originally reserved only for mathematical minds, but the application of fractional calculus motivated researchers to work on different methods for solving fractional differential or integral equations, especially to develop numerical methods. calculus deals with derivatives or integrals, which are the generalization of the usual integer ordinal derivative or integral to fractional value derivatives or integrals [1].

Fractional differential and integral equations are used to model many phenomena in many different fields of science and engineering. Following that, interest in random differential and integral equations skyrocketed. A fractional calculus that takes into consideration the randomness of real-world events is necessary. In a letter to L'Hospital that year, Leibnitz pondered, Can the meaning of derivatives of integer order be transferred to derivatives of non-integer order? (1695). In a letter dated September 30, 1695, Leibnitz posed the question would the order be 1/2? to which L'Hospital replied with another question that many consider to be the origin of fractional calculus. said It will lead to a

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contradiction from which useful future conclusions can be drawn. Leibnitz asked about a fractional derivative more than 300 years ago. Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibniz, Grunwald, and Letnikov are just a few of the well-known mathematicians who have made significant contributions to this field. The fact that fractional derivatives (and integrals) are not geographically restricted is one of the many wonderful aspects about this field of study (or quantity). Therefore, this component takes precedence and effects that are not locally concentrated into account. This topic better depicts the way things actually are in the natural world. Presenting this topic as a popular topic to the science and engineering community offers a different perspective that can help in either understanding or explaining the underlying nature. Perhaps the language of nature is fractional calculus, so it's best to speak to her in that language. However, there has been a recent push in the field of fractal research to employ the fractional derivative definition as the local operator. Over the next decade, this subfield of classical integer degree calculus, which is a superset of fractional differential calculus, will see a wide variety of applications. Therefore, only the special situation of constant fractional order differentials may be used in applications, leaving the study of variable order differential integrals as a research frontier. Perhaps by the 21st century, we'll have transitioned from ordinary calculus to fractional calculus. This is why we try to stay away from complex mathematical methods. Fractional calculus has found a variety of new uses in a wide variety of disciplines recently, including but not limited to engineering, physics, finance, applied mathematics, and bioengineering.

Euler when he wrote in 1730 that when n is a positive integer, the ratio $\frac{d^n p}{dx^n}$, p a function of x, can always be expressed algebraically. Indeed, for $n \leq m$

$$\begin{aligned}\frac{d^n p}{dx^n} &= m(m-1)(m-2).....(m-n+1)x^{m-n},\\ &= \frac{m!}{(m-n)!}x^{m-n},\\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}.\end{aligned}$$

He proposed that it might be possible to interpolate if the order n of the derivative is a fraction. Fourier in 1822 started with the integral representation of f(x) given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} cos(px - pz)dp,$$

and made the following generalization:

$$\frac{d^\mu f(x)}{dx^\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} p^\mu cos(px - pz + \frac{\mu\pi}{2})dp,$$

adding that μ could take any arbitrary value whether positive or negative. In his definition, one can see that the existence of the the fractional derivative or integral depends on the convergence of the improper integrals. Ross [4, 2, 3] attributes the first application of fractional calculus to Abel. In 1823, Abel solved the integral equation

$$\frac{1}{\Gamma(\mu)}\int_0^x \frac{\phi(t)}{(x-t)^{1-\mu}}dt = f(x) \qquad 0 < \mu < 1,$$

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which arises in connection with the tautochrone problem: A bead on a frictionless wire starts from rest at some point and slides down under the influence of gravity. What should the shape of the wire be so that the amount of time it makes the bead to descend to its lowest point is independent of its starting point? Abel obtained the solution

$$\phi(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^\mu} dt.$$

According to Butzer and Westphal, [5] Abel did not actually obtain his solution using fractional calculus but merely showed that it could be written as a fractional derivative. Yet, Abel's thoughts were crucial to the growth of fractional calculus after his death. To a large extent, Liouville is credited with developing the theory that forms the basis of modern fractional calculus. He wrote several papers on the topic between 1832 and 1855. As the first definition of a fractional derivative, Liouville's work relied on the formula for differentiating an exponential function.:

$$\frac{d^m x}{dx^m} = a^m e^{ax}.$$

He considered functions which can be written as the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x},$$

and defined the derivative as

$$\frac{d^\mu f(x)}{dx^\mu} = \sum_{n=0}^{\infty} c_n a_m^\mu e^{a_n x}$$

This definition is clearly too restrictive as it depends on the convergence of the series. He also derived the formula

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_0^\infty \phi(x+\alpha)\alpha^{\mu-1} d\mu \qquad -\infty < x < \infty, \mathbb{R}(\mu) > 0,$$

If we let $\tau = x + \alpha$, then

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_x^\infty \phi(\tau)(\tau - x)^{\mu-1} d\tau.$$

This formula is what is now known as the Liouville form of fractional integration with the factor $(-1)^\mu$ being omitted. Liouville used these equations to address issues in electrodynamics, mechanics, and geometry. It's also worth noting that the fractional derivatives in both Fourier's and Liouville's formulations are integrals. Grunwald in 1867 and Letnikov in 1868 introduced what is now known as the Grunwald-Letnikov fractional derivative. Their idea was to start with the ordinary derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and apply this recursively to obtain higher-order derivatives. For example, the second-order derivative would be:

$$f(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h},$$

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$$f(x) = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

In general, we have

$$f^n(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{0 \le m \le n} (-1)^m \binom{n}{m} f(x + (m-n)h),$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}.$$

By allowing n to be any real number, the Grunwald-Letnikov derivative is obtained:

$$D^\alpha f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{m=0}^{\frac{x-\alpha}{h}} (-1)^m \binom{\alpha}{m} f(x - mh).$$

Riemann developed his theory of fractional calculus while he was a student but it was published posthumously in 1876. Riemann sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} f(t)dt + \psi(x).$$

He felt the need to add the complementary function $\psi(x)$ to deal with the ambiguity of the lower limit of integration c, which only created even confusion as to what is meant by it. The detail study about the fractional development of differential and integral calculus can be found in [6, 7, 8, 9] and since the finding the analytically solution for the fractional order is not that simple so numerical approach to find the approximate solution is one of the alternate and simple method, several researcher work on numerical methods [10, 11, 12, 13, 14, 15], The numerical methods for solving analytical differential equations mainly classified into two categories

2 One step methods

Because only one measurement of the solution from the previous step is employed in the computation of solution in single-step techniques, they are particularly beneficial when adjusting the step size of the integration process and solution behaviour requires greater flexibility.

3 Multi-Step Methods

With multi-step approaches, the solution is often computed by iteratively applying a set of approximations that have already been tested and found to be accurate. By analogy with the standard multi-step approach to solving ordinary differential equations, the multi-step approach to solving fractional-order differential or integral equations is more precise than the one-step approach. A variety of multi-step approaches exist for resolving fractional differential equations; we provide a quick overview of some of the most common ones below.

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3.1 Product Integration rule

This rule first introduced by Young [16] in 1954 to numerically solve second-kind weakly-singular Volterra Integral Equations and since Fractional differential equations involve integration so this method is applicable to fractional differential equations also.

Given a grid $t_n = t_0 + nh$, with constant step-size $h > 0$, in product Integration rules, the solution of the volterra integral equation

$$y(t) = T_{m-1}[y; t_0](t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) \mathrm{d}\tau$$

at t_n is first written in a piece-wise way:

$$y(t_n) = T_{m-1}[y; t_0](t_n) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\kappa-1} f(\tau, y(\tau)) \mathrm{d}\tau$$

and $f(\tau, y(\tau))$ is approximated, in each subinterval $[t_j, t_{j+1}]$, by means of some interpolant polynomial; the resulting integrals are hence evaluated in an exact way, thus to lead to y_n. According to the way in which the approximation is made, explicit or implicit rules are obtained this we can extend to FDEs the (explicit) forward and (implicit) backward Euler methods, by approximating , in each interval $[t_j, t_{j+1}]$, the integrand $f(\tau, y(\tau))$ by the constant values $f(t_j, y_j)$ and $f(t_{j+1}, y_{j+1})$, respectively; the resulting methods are:
Expl. PI Rectangular: $y_n = T_{m-1}[y_j t_0](t_n) + h^a \sum_{j=0}^{n-1} b_{n-j-1}^{(a)} f(t_j, y_j)$
and:
Impl. PI Rectangular : $y_n = T_{m-1}[y_i t_0](t_n) + h^a \sum_{j=1}^{n} b_{n-j}^{(a)} f(t_j, y_j)$
with $b_n^{(a)} = ((n+1)^n - n^a) / \Gamma(\alpha+1)$; the term rectangular comes after the underlying quadrature rules used for the integration. In a similar way, when $f(\tau, y(\tau))$ is approximated by the first order interpolant polynomial:

$$f(\tau, y(\tau)) \approx f(t_{j+1}, y_{j+1}) + \frac{\tau - t_{j+1}}{h} \left(f(t_{j+1}, y_{j+1}) - f(t_j, y_j) \right), \quad \tau \in [t_j, t_{j+1}],$$

one obtains a generalization (of implicit type) of the standard trapezoidal rule:

$$\text{Impl. PI Trap. : } y_n = T_{m-1}[y_j; t_0](t_n) + h^a \left(a_n^{(a)} f_0 + \sum_{j=1}^{n} a_{n-j}^{(a)} f(t_j, y_j) \right)$$

with:

$$a_n^{(\alpha)} = \frac{(n-1)^{\alpha+1} - n^\alpha(n-\alpha-1)}{\Gamma(\alpha+2)}, \quad a_n^{(\alpha)} = \begin{cases} \frac{1}{\Gamma(\alpha+2)} \\ \frac{(n-1)^{a+1} - 2n^{k+1} + (n+1)^{\alpha+1}}{\Gamma(\alpha+2)} n = 1, 2, \ldots \end{cases}$$

These equations converges with order one and further generalized using trapezoidal rule to get more approximate solutions.

3.2 Fractional Linear Multi-Step Method

Fractional Linear Multi-Step Method first introduced by Lubich in [17], in this method he generalized quadrature rules obtained from standard linear multi-step methods because of this, the method is one of the most powerful methods for solving FDEs numerically. Given the initial value problem:

$$y(t)=f(t), \quad y\left(t_0\right)=y_{0r}$$

its solution can be approximated by

$$\sum_{j=0}^k \rho_j y_{n-j}=h \sum_{j=0}^k \sigma_j f\left(t_{n-j}\right).$$

where $\rho(z)=\rho_0 z^k+\rho_1 z^{k-1}+\cdots+\rho_k$ and $\sigma(z)=\sigma_0 z^k+\sigma_1 z^{k-1}+\cdots+\sigma_k$ are the first and second characteristic polynomial of the LMM.
The above Problem can be rewritten in the integral form:

$$y(t)=y_0+\int_{t_0}^t f(\tau) \mathrm{d} \tau$$

and its solution $y(t)$ can be approximated by using LMMs reformulated in terms of convolution quadrature formulas:

$$y_n=h \sum_{j=0}^n \omega_{n-i} f\left(t_i\right), \quad n \geq k$$

where the weights w_n depend on the characteristic polynomials $\rho(z)$ and $\sigma(z)$, but not on h. The computation of the weights w_n is usually not easy,

$$\delta(\xi)=\sum_{n=0}^{\infty} \omega_n \zeta^*, \quad \delta(\xi)=\frac{\rho(1 / \zeta)}{\sigma(1 / \zeta)}.$$

using these weights given by the coefficients of the FPS of the function:

$$F\left(\frac{\delta(\xi)}{h}\right)=\left(\frac{\delta(\xi)}{h}\right)^{-a}=h^*\left(\frac{\rho(1 / \zeta)}{\sigma(1 / \zeta)}\right)^a,$$

being $F(s)=s^{-a}$ the Laplace transform of the kernel $t^{a-1} / \Gamma(a)$. The assumptions that make possible this generalization of LMMs are that the generating function $\delta(\xi)$ has no zeros in the closed unit disc $|\xi| \leq 1$, except for $\xi=1$, and $\mid \arg \delta(\xi) \mid<\pi$ for $|\xi|<1$. LMMs satisfying these assumptions are, for instance, the backward differentiation formulas (BDFs) and the trapezoidal rule.

3.3 Generalized Adams methods

The Adams method generalized by Aceto [18] by using a product quadrature rule for Volterra integral equations with weakly singular kernels. Numerical solvers for fractional differential equations are represented by this generalisation formula, which inherits the linear stability qualities known for the integer order case. The method's useful properties have been validated by numerical studies. This technique describes and explores an

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extension of the generalised Adams methods to the fractional order case. Here, the generalisation is appealing for tough problems thanks to its linear stability qualities, as the established barrier is overcome at the cost of computing approximations of the endpoint values of the solution. Using a block implementation of the schemes (such as the Generalized Adams Methods for ODEs), this method can also be utilised to build dependable codes with automatic step size selection.

3.4 Extension of the Runge-Kutta methods

It is based on the Runge-Kutta approximation of linear multiterm fractional differential equations using convolution quadratures and includes formulas for the efficient inversion of the Laplace transform. This is an expansion of the Runge-Kutta approach developed by Garrappa citegar. In this enhancement, we address the implementation challenges and demonstrate the extension's good stability qualities; we also conduct a numerical experiment analysis of the algorithm's performance. To solve linear multiterm FDEs with good accuracy, this Runge-Kutta Extension approach competes favourably with others.. To achieve the same goal with multistep methods is a somewhat unfeasible task; indeed, good stability properties are no more assured by higher order methods of multistep types, and instability phenomena could prevail on accuracy

3.5 Generalized exponential integrator

In the process to form a numerical method to solve the Bagley-Torvik equation using the EIs method, Esmaeili [20] numerically solved the problem via a combination of the fractional exponential Euler scheme and the fractional exponential trapezoidal scheme Since theB agley-Torvik equation is a multi-term FDE, it can be solved efficiently with the proposed method. Problems with implementation and analysis of errors have been addressed. Several numerical studies have confirmed the method's accuracy and validity. Further Garrappa [21] in his paper also generalized the exponential integrator by comparing two methods of this kind and the accuracy and stability are investigated. In his method some numerical experiments are presented to validate the theoretical finding

3.6 spectral methods

Zayernouri et al [22] introduced a novel family of interpolants, dubbed fractional Lagrange interpolants, which satisfy the Kronecker delta property at collocation points, and used this to construct an exponentially accurate fractional spectral collocation approach for solving steady-state and time-dependent fractional PDEs. They use this strategy to build their own spectral theory for solving fractional SturmLiouville eigen issues. After solving a large number of linear FODEs and FPDEs, both linear and nonlinear, they extract the relevant fractional differentiation matrices and examine the numerical performance of the fractional collocation approach. In the first part, they look at the space-fractional advection-diffusion issue and generalised space-fractional multiterm first-order differential equations. Then, we move on to the solution of several FPDEs, such as the time- and space-fractional advection-diffusion equation, time- and space-fractional multiterm FPDEs, and ultimately the space-fractional Burgers equation. It has been shown numerically that the fractional collocation approach converges exponentially fast.

In continuation Zayernouri et al [23] introduced the spectral method to solve different fractional differential equations.

3.7 spectral collocation methods

In this method Burrage et al [24] to reach the numerical solution, a mixed method is proposed, which comprises of a finite difference scheme in space and a spectral collocation method in time, and is used to a general class of diffusion problems in which the standard time derivative is replaced by a fractional one. When compared to sequential approaches to discretizing the fractional derivative, the computational cost of the spectral method is significantly lower. Numerical findings are presented based on time-dependent reaction-diffusion equations, and a few classes of spectral bases are investigated, each of which has a unique convergence rate.

3.8 Methods based on matrix functions

There are number of methods based on matrix functions, Garappa [25] in his method presents a deep analysis of a time-dependent Schrdinger equation with fractional time derivative. The equation is rewritten in terms of a unique system of fractional differential equations following a discretization of the spatial operator; it so happens that the eigenvalues of the coefficient matrix lie on the edge of the stability zone for fractional differential equations. To tackle the numerical computation with different and appropriately chosen methods, the exact solution is decomposed into two or three terms (depending on the order of the fractional derivative) using theoretical analysis. This allows for the direct evaluation of matrix functions for the terms characterised by smooth behaviour but with persistent memory, and a step-by-step strategy, in conjunction with a matrix function, for the oscillating term. In both situations, we show convergence findings for computations involving matrices using Krylov subspace methods. An interesting result on a connection between the decay behaviour of its entries and the short memory principle from fractional calculus is found in a study of the matrix produced by Popolizio's (cite pop) use of the Riesz fractional derivative operator and its discretization by fractional centred differences. After that, the partial differential equation is approximated by solving for the matrix exponential of a vector that represents the provided beginning circumstances using the shift-and-invert technique. Numerical findings validate the high quality of the approximation and provide support for the suggested method. Popolizio then used matrix functions to further develop these techniques.

4 Numerical methods algorithms in Computer

Some of the codes and algorithms developed by various mathematicians make use of inverse transform methods to solve problems involving fractional differential equations, however this is not the case for ordinary or partial differential equations. A few of them join our ranks here.

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4.1 Stehfest Method

The Stehfest inversion formula [27] uses finite differences and the Salzer summation procedure to calculate a sample of the time function. It can be approximated using the form below.:

$$\hat{\mathrm{f}}(t) = \left[s \cdot \sum_{n=1}^{N} K_m \cdot F(ns) \right]_{s=\ln(2/t)}$$

where the weight coefficients K_n are given by

$$K_n = (-1)^{n+N/2} \cdot \sum_{k=(n+1)/2}^{\min(n,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(n-k)!(2k-n)!}.$$

The available parameters are the choice of N (which must be even) and the values of s at which $F(s)t$ depended must be sampled.

4.2 Abate and Whitt Method

In this method [28] the Trapezoidal rule applied to Laplace inversion formula. By applying the Trapezoidal rule with step size h and setting $h = \pi/(2t)$ and $a = A/(2t)$, we get

$$\hat{f}(t) = \frac{e^{\Lambda/2}}{2t} Re\left(F\left(\frac{A}{2t}\right)\right) + \frac{e^{\lambda/2}}{t} \sum_{k=1}^{\infty} (-1)^k Re\left(F\left(\frac{A+2k\pi}{2t}\right)\right).$$

The choice of A has to be made in such a way that a falls at the left of the real part of all singularities of the inverted Laplace transform.

4.3 De Hoog Method

Numerical inversion of Laplace transforms is enhanced using this new method. From the use of non-linear double acceleration with Pad approximation and an analytic equation for the remainder in the series, we can hasten the convergence of the Fourier series produced through the inversion of integral. The number of function evaluations can be reduced by a large margin with non-linear acceleration methods. The order of the Taylor series expansion, which governs the algorithm's speed, is set by the user.[29].

4.4 FFT Method

The non-accelerated Fouricr series inverse algorithm is almost useless because it requires thousands of F(s) evaluations. Practical approaches accelerate the convergence of the sum by using for example well known FFT (Fast Fourier Transform) algorithm [30].

The program inverts numerically a Laplace transform $f(s)$ into $f(t)$ using the Fast Fouricr Transform (FFT) algorithm for a specific time t, an upper frequency limit ω, a real parameter σ and the number of integration intervals N. Parameter σ contains singularities coordinates of F(s) and must be input by a user.

4.5 Vlach and Singhai Method

The basic inversion formula which works for real time functions assumes the following form [31]

$$\hat{\mathrm{f}}(t) = -\frac{1}{t}\sum_{i=1}^{M} K_i F\left(\frac{z_i}{t}\right),$$

where K_i are coefficients obtained as a result of Padce approximation of exponential part of the Laplace inversion formula.

4.6 Zakian Method

Zakian developed a method [32], in which approximation for $f(t)$ is given by

$$\hat{\mathrm{f}}(t) = \frac{1}{t}\sum_{i=1}^{N} K_i F\left(\frac{\alpha}{t}\right)$$

where sets of $\alpha_i K_i$ can be obtained as solution to

$$\sum_{i=1}^{N} \frac{K_i k!}{(\alpha_i)^{k+1}} = 1, k = 0, 1, 2, \ldots .2N - 1.$$

The only free parameter of choice is N. Its value should be 10 for best results.

4.7 Talbot Method

Talbot developed a method for the numerical inversion of the Laplace transform, in which the inversion is approximated by the Trapezoidal rule along a special deformed contour. The Inverse Laplace Transform formula can be rewritten as

$$\hat{f}(t) = \frac{1}{2\pi i}\oint_{\mathbf{B}} F(s)e^{st}ds,$$

where **B** is the Bromwich contour from $r - j\infty$ to $r + j\infty$ and $t > 0$, so that B is to the right of all singularities of $F(s)$.

Talbot provided [33] a recipe for a contour, in which the contour is moved to the left so as to reduce in magnitude the factor e^{st} in the integrand, but the contour must not be moved too close to singularities of $F(s)$, as to do so will result in peaks in the integrand function. This requires to be known the locations of the singularities of $F(s)$.

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