Review on Different Differential and Integral Operators in Fractional Calculus

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Abstract

Differential and integral operators connected with the solution of fractional differential equations and fractional integral equations are discussed in this article. Since there isn't a single operator that unites the features, the calculus of arbitrary orders, and fractional orders in particular, is a very antiquated field, and we quickly go over the numerous types of differential and integral operators that can be used. Although there are many different operators, we aim to include the most well-known ones.

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1 Introduction

Since its humble beginnings as a mathematical curiosity during Leibniz's time, the study of fractional calculusthe area of mathematics concerned with derivatives and integrals of non-integer orderhas flourished into a vibrant area of study. Famous mathematicians laid the groundwork for the current theory, allowing fractional calculus to enter mainstream mathematics: Riemann, Liouville, Grunwald, Euler, Lagrange, Caputo, and others. This area of study is still growing at the present time. Numerous new ideas and concepts have emerged, with numerous potential applications in fields as diverse as viscoelasticity, fluid flow, rheology, etc. The roots of fractional calculus can be traced all the way back to the infancy of the field of calculus. The concept of fractions in calculus can be traced back to the age-old problem of how far one can take a phrase. Extensions of the meaning of real numbers to complex numbers and factorials of integers to complex numbers are two such examples. Many academics, however, are still unfamiliar with this area. They often ask: What is a fractional derivative? Is this a new field or an old field? Are there applications of fractional derivatives? What are those applications?

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Euler when he wrote in 1730 that when n is a positive integer, the ratio $\frac{d^n p}{dx^n}$, p a function of x, can always be expressed algebraically. Indeed, for $n \leq m$

$$\frac{d^n p}{dx^n} = m(m-1)(m-2).....(m-n+1)x^{m-n},$$
$$= \frac{m!}{(m-n)!}x^{m-n},$$
$$= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}.$$

He proposed that it might be possible to interpolate if the order n of the derivative is a fraction. Fourier in 1822 started with the integral representation of f(x) given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} cos(px - pz)dp,$$

and made the following generalization:

$$\frac{d^\mu f(x)}{dx^\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} p^\mu cos(px - pz + \frac{\mu\pi}{2})dp,$$

adding that μ could take any arbitrary value whether positive or negative. In his definition, one can see that the existence of the the fractional derivative or integral depends on the convergence of the improper integrals. Ross [11, 12, 13] attributes the first application of fractional calculus to Abel. In 1823, Abel solved the integral equation

$$\frac{1}{\Gamma(\mu)}\int_0^x \frac{\phi(t)}{(x-t)^{1-\mu}}dt = f(x) \qquad 0 < \mu < 1,$$

which arises in connection with the tautochrone problem: A bead on a frictionless wire starts from rest at some point and slides down under the influence of gravity. What should the shape of the wire be so that the amount of time it makes the bead to descend to its lowest point is independent of its starting point? Abel obtained the solution

$$\phi(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^\mu}dt.$$

According to Butzer and Westphal, [1] Abel just demonstrated that his solution could be expressed as a fractional derivative; he did not use fractional calculus to arrive at his answer. Yet, Abel's thoughts were crucial to the growth of fractional calculus after his death. To a large extent, Liouville is credited with developing the theory that forms the basis of modern fractional calculus. He wrote several papers on the topic between 1832 and 1855. As the first definition of a fractional derivative, Liouville's work relied on the formula for differentiating an exponential function:

$$\frac{d^m x}{dx^m} = a^m e^{ax}.$$

He considered functions which can be written as the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x},$$

and defined the derivative as

$$\frac{d^\mu f(x)}{dx^\mu} = \sum_{n=0}^{\infty} c_n a_m^\mu e^{a_n x}$$

This definition is clearly too restrictive as it depends on the convergence of the series. He also derived the formula

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_0^\infty \phi(x+\alpha)\alpha^{\mu-1} d\mu \qquad -\infty < x < \infty, \mathbb{R}(\mu) > 0,$$

If we let $\tau = x + \alpha$, then

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_x^\infty \phi(\tau)(\tau - x)^{\mu-1} d\tau.$$

This formula is what is now known as the Liouville form of fractional integration with the factor $(-1)^\mu$ being omitted. Liouville applied these formulas to solve various problems in electrodynamics, mechanics and geometry. It is also worthwhile to note that in both Fourier's and Liouville's definitions, the fractional derivatives take the form of an integral. Grunwald in 1867 and Letnikov in 1868 introduced what is now known as the Grunwald-Letnikov fractional derivative. Their idea was to start with the ordinary derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and apply this recursively to obtain higher-order derivatives. For example, the second-order derivative would be:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h},$$
$$f''(x) = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

In general, we have

$$f^n(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{0 \le m \le n} (-1)^m \binom{n}{m} f(x + (m-n)h),$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}.$$

By allowing n to be any real number, the Grunwald-Letnikov derivative is obtained:

$$D^\alpha f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{m=0}^{\frac{x-\alpha}{h}} (-1)^m \binom{\alpha}{m} f(x - mh).$$

Riemann's theory of fractional calculus was developed during his time as a student, although it was not published until after his death in 1876. While looking for a generalisation of a Taylor's series expansion, Riemann came up with the following definition of fractional integration:

$$\frac{d^{-\mu}f(x)}{dx^{-\mu}} = \frac{1}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1}f(t)dt + \psi(x).$$

He felt the need to add the complementary function $\psi(x)$ to deal with the ambiguity of the lower limit of integration c, which only created even confusion as to what is meant by it. The detail study about the fractional development of differential and integral calculus can be found in [1, 2, 3, 4] and since the finding the analytically solution for the fractional order is not that simple so numerical approach to find the approximate solution is one of the alternate and simple method, several researcher work on numerical methods [5, 6, 7, 8, 9, 10].

2 Fractional differential and integral operators

As the number of differential and integral operators in fractional calculus grows, it becomes sense to compile them all in one place and examine their categorization to see if they meet the requirements for the fractional order. Since integrals are used to define the vast majority of fractional calculus' differential operators, we've decided to group them together into a single category rather than categorise them independently.

2.1 Riemann-Liouville Left sided and right-sided operator

Riemann-Liouville define the most popular fractional operator [11, 12, 13] in the form of left-sided operator

$$^{RL}\mathrm{D}^{\alpha}_{a^+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x (x-\xi)^{n-\alpha-1}f(\xi)\mathrm{d}\xi, x \geq a$$

in the similar way the right-sided introduced was

$$^{RL}\mathrm{D}^{\alpha}_{b^-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b (\xi-x)^{n-\alpha-1}f(\xi)\mathrm{d}\xi, x \leq b$$

2.2 Caputo left-sided and right sided operator

The next extended fractional operator [13, 14, 15, 16] is the Caputo left sided operator

$$^{C}\mathrm{D}_{\mathrm{a}^+}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\xi)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\mathrm{d}\xi, x \geq a$$

and the right sided one is

$$ {}^{c}\mathrm{D}_{b}^{\alpha}[f(x)]=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}(\xi-x)^{n-\alpha-1}\frac{\mathrm{d}^{n}}{d\xi^{n}}[f(\xi)]\mathrm{d}\xi, x\leq b $$

2.3 Weyl fractional operator

Weyl define the fractional derivative using the operator [17]

$$ {}_{x}\mathrm{D}_{\infty}^{\alpha}[f(x)]=\mathrm{D}_{-}^{\alpha}[f(x)]=(-1)^{\mathrm{m}}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{n}\left[{}_{x}\ \mathrm{W}_{\infty}^{\alpha}[f(x)]\right] $$

with $\left[{}_{x}\ \mathrm{W}_{\infty}^{\alpha}[f(x)]\right]=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\mathrm{a}-1}f(t)\mathrm{d}t.$

2.4 Marchaud Fractional operator

The further generalized fractional operator [17] by Marchaud is

$$ D_{+}^{\alpha}[f(x)]=\frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{f(x)-f(\xi)}{(x-\xi)^{1+\alpha}}\mathrm{d}\xi $$

2.5 Hadamard Fractional operator

Hadamard also defined the fractional differential operator[18, 19] as

$$ D_{+}^{a}[f(x)]=\frac{x}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\left(\ln\frac{x}{\tau}\right)^{2-a}f(\tau)\frac{\mathrm{d}\tau}{\tau} $$

2.6 Chen Fractional operator

Chen [20] introduced the fractional operator as

$$ \mathrm{D}_{\mathrm{c}}^{\alpha}[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{c}^{x}(x-\xi)^{-\alpha}f(\xi)\mathrm{d}\xi, x\geq c $$

2.7 Davidson-Essex Fractional operator

The Davidson-Essex fractional operator is as follows

$$ \mathrm{D}_{0}^{\alpha}[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}^{n+1-k}}{\mathrm{d}x^{n+1-k}}\int_{0}^{x}(x-\xi)^{-\alpha}\frac{\mathrm{d}^{k}}{\mathrm{d}\xi^{k}}[f(\xi)\mid\mathrm{d}\xi $$

2.8 Canavati Fractional operator

The Fractional differential operator by Canavati is

$$ {}_aD_x^v[f(x)] = \frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^\mu \frac{\mathrm{d}^n}{\mathrm{d}^n}[f(\xi)]\mathrm{d}\xi, n = \lfloor \nu \rfloor, \mu = n - v $$

2.9 Jumarie Fractional operator

Jumarie [21, 22, 23] next extended Fractional operator as

$$ \mathrm{D}_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x (x-\xi)^{n-\alpha-1}[f(\xi)-f(0)]\mathrm{d}\xi $$

2.10 Erdlyi-Kober Fractional operator

One of the most used fractional operator was defined by Erdlyi-Kober derivative [24]

$$ D_{a+;\sigma,\eta}^\alpha f(x) = x^{-\sigma(\alpha+\eta)}\left(\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x^{\sigma(\alpha+n+\eta)} I_{a+;\sigma,\eta}^{\alpha+n} f(x), \alpha > -n, $$

with

$$ I_{a+;\sigma,\eta}^\alpha f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x f(\tau)\left(x^\sigma - \tau^\sigma\right)^{\alpha-1}\tau^{\sigma\eta+\sigma-1}\,\mathrm{d}\tau, \alpha > 0 $$

and

$$ D_{b-;\sigma,\eta}^\alpha f(x) = x^{\sigma\eta}\left(-\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x^{\sigma(n-\eta)} I_{b-;\sigma,\eta-n}^{\alpha+n} f(x), \alpha > -n, $$

with

$$ I_{b-;\sigma,\eta}^\alpha f(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int_x^b f(\tau)\left(\tau^\sigma - x^\sigma\right)^{\alpha-1}\tau^{\sigma(1-\alpha-\eta)-1}\,\mathrm{d}\tau, \alpha > 0 $$

2.11 Regularized Liouville Fractional Operator

Another operator defined [25] as follows

$$ D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \tau^{-\alpha-1}\left[f(t-\tau) - \sum_{m=0}^{N-1}\frac{(-1)^m f^{(m)}(t)}{m!}\tau^m\right]\mathrm{d}\tau $$

with $N = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ the integer part of α.

2.12 Riesz/Feller Fractional Operator

The Riesz/Feller fractional operator [25] is

$$D_\theta^\alpha f(t)=\frac{1}{2\sin(\alpha\pi)\Gamma(-\alpha)}\int_{\mathrm{R}} f(t-\tau)\sin\left[(\alpha+\theta\cdot\operatorname{sgn}(\tau))\frac{\pi}{2}\right]|\tau|^{-\alpha-1}\,\mathrm{d}\tau,$$

with $\theta\in\mathbb{R}$ and $\operatorname{sgn}(\cdot)$ denoting the signal function.

The two sided operator is as follows

$$D_{\mathrm{C}}^\gamma f(t)=\lim_{h\to 0^+} h^{-\gamma}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{\Gamma(\gamma+1)}{\Gamma\left(\frac{\gamma+\theta}{2}-n+1\right)\Gamma\left(\frac{\gamma-\theta}{2}+n+1\right)}f(t-nh),$$

with $\gamma>-1$.

2.13 Hilfer Fractional Operator

Hilfer defined the fractional operator [26] as

$$D_{a\pm}^{\alpha;\mu}f(t)=\pm I_{a\pm}^{\mu(1-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)I_{a\pm}^{(1-\mu)(1-\alpha)}f(t),0\le\mu\le 1,$$

where $0<\alpha<1$ and

$$I_{a+}^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)(t-\tau)^{\alpha-1}\,\mathrm{d}\tau, t\ge a$$

$$I_{b}^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b f(\tau)(\tau-t)^{\alpha-1}\,\mathrm{d}\tau, t\le b.$$

2.14 *k*-Hilfer Fractional Operator

The K-generalization of Hilfer fractional operator [27] is

$${}^kD^{\mu,\nu}f(t)=I_k^{v(1-\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)I_k^{(1-\mu)(1-v)}f(t),0\le\mu\le 1,$$

where $0<\nu<1$ and

$$I_k^\alpha f(t)=\frac{1}{k\Gamma_k(\alpha)}\int_0^t f(\tau)(t-\tau)^{\frac{\alpha}{k}-1}\,\mathrm{d}\tau, t\ge 0.$$

2.15 Hilfer-Katugampola Fractional Operator

Another generalization of Hilfer operator is Hilfer-Katugampola [28] as

$${}^\rho D_{a\pm}^{\alpha,\beta}f(x)=\left[\pm{}^\rho I_{a\pm}^{\beta(1-\alpha)}\left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)\rho I_{a\pm}^{(1-\beta)(1-\alpha)}\right]f(t),\rho>0,$$

where $0 < \alpha < 1, 0 \leq \beta \leq 1$ and

$$\rho_{a+}^{\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^z f(\tau) \left(x^\rho - \tau^\rho\right)^{\alpha-1} \mathrm{d}\tau, x > a$$
$$ {}^\rho I_{b-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_z^b f(\tau) \left(\tau^\rho - x^\rho\right)^{\alpha-1} \mathrm{d}\tau, x < b.$$

2.16 ψ-Hilfer Fractional Operator

The Hilfer operator involving the special function ψ known as ψ-Hilfer[?] is as as follows

$$H\mathrm{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x), 0 \leq \beta \leq 1,$$

and

$$H_{\mathbb{D}_{b-}^{\alpha,\beta;\psi}} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x), 0 \leq \beta \leq 1.$$

Here $n - 1 < \alpha < n$ with $n \in \mathbb{N}, I = [a, b]$ is an interval such that $-\infty \leq a < b \leq \infty$, ψ denotes an increasing function such that $\psi'(x) \neq 0$ for all $x \in I$ and $\psi \in C^n([a, b], \mathbb{R})$. The corresponding integral (on the left and on the right) are given by

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)[\psi(x) - \psi(t)]^{\alpha-1} f(t) \mathrm{d}t$$

and

$$I_{b-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_z^b \psi'(t)[\psi(t) - \psi(x)]^{\alpha-1} f(t) \mathrm{d}t$$

respectively.

Numerous additional functions have been constructed to accommodate the fractional order treatment of differentiation and integration, and there are still a great many Fractional differential and Integral operators and further generalisation to derivatives and integrals. The definitions given here are the most commonly used ones in fractional calculus.

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