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Geodesic equations of Randers-change of Matsumoto metric in two-dimensional Finsler spaces

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Abstract

Geodesics are locally length minimizing curves. In the study of geometry equation of geodesics plays a vital role. In the present paper, we find the geodesic equation of Randers-change of Matsumoto metric and special (α, β)-metric of two dimensional Finsler space.

Keywords: Finsler space, Randers metric, Matsumoto metric, Geodesic equations, $(\boldsymbol{\alpha, \beta})$-metric.

1. Introduction

A differential equation of geodesics plays an important role in two-dimensional Finsler space, which is an intriguing research area. It was studied by many research experts. Firstly, in 1994 work was started by Matsumoto [6] and after that Matsumoto collaborated his further work with H. S. Park [9] in 1997 they established the equation of geodesics in two-dimensional Finsler spaces with the Randers metric and the Kropina metric and in 1998, they have extended same work with the slope metric (i.e., $L = \frac{\alpha^2}{\alpha-\beta}$), β is the infinitesimal of the degree one, if it has degree more than two which is neglecting, they obtained the equations of geodesic of two-dimensional Finsler space in the form of $y'' = f(x, y, y')$, where (x, y) are the co-ordinates of two-dimensional Finsler spaces [9]. In 2000, Park has studied the further work with Lee [11] across generalized Kropina metric for the above case. Next in 2013, V. K. Chaubey continued the work with his co-researchers [3-4] B. N. Prasad and D. D. Tripathi, they all together worked on different special (α, β)-metrics beneath the same conditions.

Matsumoto introduced the concept of Randers change of Finsler metrics [7]. Later, using this, Park and Lee [12] studied the condition for Finsler spaces with (α, β) metrics transformed by Randers change to be of Douglas type. In this present paper, we considered the Randers change of Matsumoto metric $L = \frac{\alpha^2}{\alpha-\beta} + \beta$ and $L = \alpha + \sqrt{\alpha^2 + \beta^2}$ to find the equation of geodesics in two-dimensional Finsler space. In the next steps this article set out like this which follows: In part 2, we determined the preliminaries to obtain the geodesics with (α, β)- metric in two-dimensional Finsler space. In part 3, we go through the ideas of solving the equation of geodesics. Finally, in parts 4 and 5 we get the equation of geodesics in two-dimensional space with Finsler metrics and described the main result, is presented in theorem.

2. Preliminaries

Let M be an n-dimensional Finsler space i.e., M^n. TM be the bundle of non-zero tangent vectors of a smooth manifold. Here we have a formal definition of Finsler space as follows:

Definition 2.1. [4] A Finsler space is a triple $F^n = (M, D, L)$, where M is an n-dimensional manifold, D is an open subset of a tangent bundle TM and L is a Finsler metric defined as a function $L: TM \to [0,1)$ with the following properties:

1. Regular: L is C^∞ on the entire tangent bundle $TM\backslash\{0\}$.
2. Positive homogeneous: $L(x, \lambda y) = \lambda\, L(x, y),\ \lambda > 0$.
3. Strong convexity: The $n \times n$ Hessian matrix $g_{ij}(x, y) = \frac{1}{2}[L^2]y^i y^j$ is positive definite at every point on $TM\backslash\{0\}$, where $TM\backslash\{0\}$ denotes the tangent vector y is nonzero in the tangent bundle TM.

Now, we consider a two-dimensional Finsler spaces, $F^2 = (M^2, L)$ with Finsler metric function $L(x^1, x^2; y^1, y^2)$. We denote $\frac{\partial f}{\partial x^i} = f_i$, $\frac{\partial f}{\partial y^i} = f_{(i)}, (i = 1,2)$ for any Finsler function $f(x^1, x^2; y^1, y^2)$. Here the suffices i, j run over 1, 2. Since $L(x^1, x^2; y^1, y^2)$ is (1) p-homogeneous in (y^1, y^2), we have $L_{(j)(i)}y^i = 0$, which imply the existence of a function, the Weierstrass invariant $W(x^1, x^2; y^1, y^2)$ [9, 11] is given by

$$\frac{L_{(1)(1)}}{(y^2)^2} = -\frac{L_{(1)(2)}}{y^1 y^2} = \frac{L_{(2)(2)}}{(y^1)^2} = W(x^1, x^2; y^1, y^2) \tag{2.1}$$

In a two-dimensional Finsler space, associated Riemannian space $R^2 = (M^2, \alpha)$ with respect to $L = \alpha$ and $= \sqrt{a_{ij}y^i y^j}$, the Weierstrass invariant W_r of R^2 is written as

$$W_r = \frac{1}{\alpha^3}\{a_{11}a_{22} - (a_{12})^2\}$$

Further L_j are still (l) p-homogeneous in (y^1, y^2), so that we get

$$L_{j(i)}y^i = L_j \tag{2.2}$$

The geodesic equations in two-dimensional Finsler spaces F^2 along with curve $C: x^i = x^i(t)$ are given by [9]

$$L_i - \frac{dL_i}{dt} = 0 \tag{2.3}$$

Substituting Eq. (2.2) in Eq. (2.3), we get

$$L_{1(2)} - L_{2(1)} + (y^1\dot{y}^2 - y^2\dot{y}^1)W = 0 \tag{2.4}$$

which is called the Weierstrass form of geodesic equation in F^2, [2, 9] where $\dot{y}^i = \frac{dy^i}{dt}$. For the metric function $L(x, y, \dot{x}, \dot{y})$ and Eq. (2.4) becomes to

$$\frac{\partial^2 L}{\partial\dot{y}\,\partial x} - \frac{\partial^2 L}{\partial\dot{x}\,\partial y} + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\frac{\partial^2 L}{(\partial\dot{y})^2} = 0 \tag{2.5}$$

Let $\Gamma = \{\gamma^i_{jk}(x^1, x^2)\}$ be the Levi-civita connection of the associated Riemannian space R^2. We introduce the linear Finsler connection $\Gamma^* = (\gamma^i_{jk}, \gamma^i_{0j}, 0)$ and the h- and v-covariant differentiation in Γ^* are denoted by $(;i,(i))$ respectively, where the index (0) means the contraction with y^i, then $y^i_{;j} = 0, \alpha_{;i} = 0$ and $\alpha_{(i);j} = 0$.

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3. Geodesic equations of special (α, β)-metric

In this part, we consider the two-dimensional Finsler spaces $F^2 = \left(M^2, L(\alpha, \beta)\right)$, where $\beta = b_i(x^1, x^2)y^i$ and discuss the geodesic equation [12], we have

$$L_{;i} = L_\beta \beta_{;i},\ L_{(i)} = L_\alpha \alpha_{(i)} + L_\beta b_{(i)} \tag{3.1}$$

where $\alpha_{(i)} = \frac{a_{ir}y^r}{\alpha}$ and the subscriptions α, β of L are the partial derivatives of L with respect to α, β respectively. Then we have in Γ^*.

$$L_{(j);i} = L_{(j)i} - L_{(j)(r)}{\gamma_0}_i^r - L_{(r)}{\gamma_j^r}_i$$

from which

$$L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + L_{(2)(r)}{\gamma_0}^r{}_1 - L_{(1)(r)}{\gamma_0}^r{}_2 \tag{3.2}$$

From Eq. (2.1) and Eq. (3.2) we have

$$L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + \left(y^1{\gamma_0}^2{}_0 - y^2{\gamma_0}^1{}_0\right)W \tag{3.3}$$

On other hand, from Eq. (3.1) we have

$$L_{(j);i} = L_{\alpha\beta}\beta_{;i}\alpha_{(j)} + L_{\beta\beta}\beta_{;i}b_j + L_\beta b_{j;i} \tag{3.4}$$

Similarly to the case of $L(x^1, x^2; y^1, y^2)$ and $\alpha(x^1, x^2)$, we get the Weierstrass invariant $w(\alpha, \beta)$ as follows:

$$w = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2} \tag{3.5}$$

Substituting Eq. (3.5) in Eq. (3.4), we have

$$L_{(j);i} = \alpha w \beta_{;i}\left(\alpha b_j - \beta\alpha_{(j)}\right) + L_\beta b_{j;i} \tag{3.6}$$

From Eq. (3.4) and Eq. (3.6) we have

$$L_{1(2)} - L_{2(1)} = \alpha w\left\{\beta_{;1}\left(\alpha b_2 - \beta\alpha_{(2)}\right) - \beta_{;2}\left(\alpha b_1 - \beta\alpha_{(1)}\right)\right\} - L_\beta\left(b_{1;2} - b_{2;1}\right) + \left(y^1{\gamma_0}^2{}_0 - y^2{\gamma_0}^1{}_0\right)W. \tag{3.7}$$

If we put ${y^i}_{;0} = \dot{y}^i + {\gamma_0}^i{}_0$, we get

$$y^1\dot{y}^2 - y^2\dot{y}^1 = y^1y^2_{;0} - y^2y^1_{;0} - \left(y^1{\gamma_0}^2{}_0 - y^2{\gamma_0}^1{}_0\right). \tag{3.8}$$

Substituting Eq. (3.7) and Eq. (3.8) in Eq. (2.4), we have

$$\alpha w\left\{\beta_{;1}\left(\alpha b_2 - \beta\alpha_{(2)}\right) - \beta_{;2}\left(\alpha b_1 - \beta\alpha_{(1)}\right)\right\} - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) + \left(y^1y^2_{;0} - y^2y^1_{;0}\right)W = 0, \tag{3.9}$$

where $\beta_{;i} = b_{r;i}y^r$. The relation of W, W_r and w is written as follows:

$$W = (L_\alpha + \alpha w\gamma^2)W_r \tag{3.10}$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $b^2 = a^{ij}b_ib_j$. Therefore Eq. (3.9) is expressed as follows:

$$\begin{aligned}(L_\alpha + \alpha w\gamma^2)\left(y^1y^2_{;0} - y^2y^1_{;0}\right)W_r - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) + \alpha w\{&\left(\alpha b_2 - \beta\alpha_{(2)}\right)\\ &-b_{0;2}\left(\alpha b_1 - \beta\alpha_{(1)}\right)\} = 0\end{aligned} \tag{3.11}$$

Thus, we have the following theorems:

Theorem 3.1. [12] In a two-dimensional Finsler spaces F^2 with an (α, β)-metric, the differential equation of a geodesic is given by Eq. (3.11).

Suppose that α be positive-definite. Then we may refer to an isothermal coordinate system $(x^i) = (x, y)$ Eq. (3.2), Eq. (3.3) such that

$$\alpha = aE, a = a(x, y) > 0, E = \sqrt{\dot{x}^2 + \dot{y}^2},$$

that is, $a_{11} = a_{22} = a^2, a_{12} = 0$ and $(y^1, y^2) = (\dot{x}, \dot{y})$. From $\alpha^2 = a_{ij}(x)y^i y^j$ we get $\alpha\alpha_{(i)(j)} = a_{ij} - a_{ir}a_{js}\frac{y^r y^s}{\alpha^2}$. Therefore, we have $\alpha\alpha_{(1)(1)} = \left(\frac{a\dot{y}}{E}\right)^2$ and $W_r = \frac{a}{E^3}$. Furthermore, the Christoffel symbols are given by

$$\gamma_1{}^1{}_1 = -\gamma_2{}^1{}_2 = \gamma_1{}^2{}_2 = \frac{a_x}{a}, \ \gamma_1{}^1{}_2 = -\gamma_1{}^2{}_1 = \gamma_2{}^2{}_2 = \frac{a_y}{a},$$

where $a_x = \frac{\partial a}{\partial x}$, $a_y = \frac{\partial a}{\partial y}$. Therefore, we have

$$\left(y^1 y^2_{;0} - y^2 y^1_{;0}\right)W_r = \frac{a}{E^3}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{1}{E}\left(a_x\dot{y} - a_y\dot{x}\right) \tag{3.12}$$

Next calculating $\gamma^2 = b^2\alpha^2 - \beta^2$, $b_{0;1}\left(\alpha b_2 - \beta\alpha_{(2)}\right)$ and $b_{0;2}\left(\alpha b_1 - \beta\alpha_{(1)}\right)$ we have

$$\gamma^2 = \{(b_1)^2 + (b_2)^2\}(\dot{x}^2 + \dot{y}^2) - (b_1\dot{x} + b_2\dot{y})^2 = (b_1\dot{y} - b_2\dot{x})^2, \tag{3.13}$$

$$b_{r;1}\left(\alpha b_2 - \beta\alpha_{(2)}\right)y^r = \frac{a}{E}b_{0;1}(b_2\dot{y} - b_1\dot{x})\dot{x}, \tag{3.14}$$

$$b_{r;2}\left(\alpha b_1 - \beta\alpha_{(1)}\right)y^r = \frac{a}{E}b_{0;2}(b_1\dot{y} - b_2\dot{x})\dot{y}, \tag{3.15}$$

substituting Eq. (3.12), Eq. (3.13), Eq. (3.14) and Eq. (3.15) in Eq. (3.11), we have

$$\begin{aligned}&\left\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2\left(a_x\dot{y} - a_y\dot{x}\right)\right\}\{L_\alpha + aEw(b_1\dot{y} - b_2\dot{x})^2\}\\ &-E^3L_\beta\left(b_{1y} - b_{2x}\right) - E^3a^2w\left(b_{1\dot{y}} - b_{2\dot{x}}\right)b_{0;0} = 0,\end{aligned} \tag{3.16}$$

where,

$$\begin{aligned}b_{0;0} = b_{rs}y^r y^s = \left(b_{1x}\dot{x} + b_{1y}\dot{y}\right)\dot{x} + \left(b_{2x}\dot{x} + b_{2y}\dot{y}\right) + \frac{1}{a}\{(\dot{x}^2 + \dot{y}^2)\left(a_x b_1 + a_y b_2\right)\\ -2(b_1\dot{x} + b_2\dot{y})\left(a_x\dot{x} + a_y\dot{y}\right)\},\end{aligned} \tag{3.17}$$

where, $b_{ix} = \frac{\partial b_i}{\partial x}$ and $b_{iy} = \frac{\partial b_i}{\partial y}$. Thus, we have the following:

Theorem 3.2. [12] In two-dimensional Finsler spaces F^2 with an (α, β)-metric, if we refer to an isothermal coordinate system (x, y) such that $\alpha = aE$, then the equation of a geodesic is given by Eq. (3.16) and Eq. (3.17).

4. Geodesic equations of Randers change of Matsumoto Metric

The Finsler metric function, $L = \frac{\alpha^2}{\alpha - \beta} + \beta$ is called Randers change of Matsumoto metric.

$$L_\alpha = \frac{\alpha(\alpha - 2\beta)}{(\alpha - \beta)^2}, \ L_\beta = 1 + \frac{\alpha^2}{(\alpha - \beta)^2}, \ w = \frac{2}{(\alpha - \beta)^3} \tag{4.1}$$

where, $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$.

Substituting all these values of Eq. (4.1) in Eq. (3.16), we get the equation of a geodesic in an isothermal coordinate system (x, y) with respect to α as follows:

$$\begin{aligned}&\left\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2\left(a_x\dot{y} - a_y\dot{x}\right)\right\}\{\alpha(\alpha - \beta)(\alpha - 2\beta) + 2\alpha(b_1\dot{y} - b_2\dot{x})^2\}\\ &-E^2a\left\{E^2(\alpha - \beta)((\alpha - \beta)^2 + \alpha)\left(b_{1y} - b_{2x}\right) + 2\alpha\left(b_{1\dot{y}} - b_{2\dot{x}}\right)b_{0;0}\right\} = 0.\end{aligned} \tag{4.2}$$

In certain case suppose for the parameter t of a curve C is chosen x of (x, y), then $\dot{x} = 1$, $\dot{y} = y'$, $\ddot{x} = 0$, $\ddot{y} = y''$, $E = \sqrt{1 + (y')^2}$ and thus

$$\left\{ay'' + (1 + (y')^2)\left(a_x y' - a_y\right)\right\}\{\alpha(\alpha - \beta)(2\alpha(\alpha - \beta) + \beta^2)\left(b_{1y} - b_{2x}\right) + 2\alpha(b_1 y^1 - b_2)b_{0;0}\} = 0, \tag{4.3}$$

where,

$$b_{0;0} = (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y' + \frac{1}{a}\{(1 + (y')^2)(a_x b_1 + a_y b_2) - 2(b_1 + b_2 y')(a_x + a_y y')\} \quad (4.4)$$

Thus, we state:

Theorem 4.1. *Let a two-dimensional Finsler spaces F^2 with a randers change of matsumoto metric. If we refer to a local coordinate system (x, y) with respect to α, then the differential equation of a geodesic $y = y(x)$ of F^2 is of the form*

$$y'' = \frac{g(x, y, y')}{f(x, y, y')}$$

where $f(x, y, y')$ is a quadratic polynomial in y' and $g(x, y, y')$ is a polynomial in y' of degree at most five.

In order to find the concrete form, we treat the case of local coordinate system, associated Riemannian space is Euclidean with orthonormal coordinate system. Then $a = 1$ and $a_x = a_y = 0$. If we fill b_1 and b_2 such that $b_1 = b_x, b_2 = b_y$ for scalar b then $b_{1y} - b_{2x} = 0$. Therefore Eq. (4.3) reduces to

$$y'' = \frac{2\alpha(1+(y\prime)^2)(b_x y\prime - b_y)b_{0;0}}{\alpha(\alpha-\beta)(\alpha-2\beta)+2\alpha(b_1 y\prime - b_2)^2}, \quad (4.5)$$

where,

$$b_{0;0} = (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y'. \quad (4.6)$$

Thus, we have the following:

Corollary 4.2. *Let F^2 be a two-dimensional Finsler space with a randers change of matsumoto metric $L = \frac{\alpha^2}{\alpha-\beta} + \beta$. If refer to an orthonormal coordinate system (x, y) with respect to α and $b_1 = \frac{\partial b}{\partial x}, b_2 = \frac{\partial b}{\partial y}$ for a scalar b, then the differential space of geodesic $y = y(x)$ of F^2 is given by Eq. (4.5).*

5. Geodesic equations of special (α, β)-metric $L = \alpha + \sqrt{\alpha^2 + \beta^2}$

Consider the special (α, β)- metric $L = \alpha + \sqrt{\alpha^2 + \beta^2}$

$$L_\alpha = \frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha^2+\beta^2}}, \quad L_\beta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}, \quad w = \frac{1}{(\alpha^2+\beta^2)^{3/2}} \quad (5.1)$$

Substituting all values of Eq. (5.1) in Eq. (3.16), we get the equation of a geodesic in an isothermal coordinate system (x, y) with respect to α as follows:

$$\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\}\{(\alpha^2 + \beta^2)\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right) + \alpha(b_1\dot{y} - b_2\dot{x})^2\}$$
$$-E^2 a\{E^2\beta(\alpha^2 + \beta^2)(b_{1y} - b_{2x}) + \alpha(b_{1\dot{y}} - b_{2\dot{x}})b_{0;0}\} = 0 \quad (5.2)$$

In the certain case suppose for the parameter t of a curve C is chosen x of (x, y), then $\dot{x} = 1, \dot{y} = y', \ddot{x} = 0, \ddot{y} = y'', E = \sqrt{1 + (y')^2}$

$$ay''\{(\alpha^2 + \beta^2)(\alpha + \sqrt{\alpha^2 + \beta^2}) + \alpha(b_1 y' - b_2)^2\} + \{(1 + (y')^2)(\alpha)(b_x y' - b_y)b_{0;0}\} = 0 \quad (5.3)$$

where,

$$b_{0;0} = (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y' + \frac{1}{a}\{(1 + (y')^2)(a_x b_1 + a_y b_2) - 2(b_1 + b_2 y')(a_x + a_y y')\} \quad (5.4)$$

Thus, we state:

Theorem 5.1. *Let a two-dimensional Finsler space F^2 with (α,β)-metric. If we mention to a local coordinate system (x,y) with respect to α, then the differential equation of a geodesic $y=y(x)$ of F^2 is of the form*

$$y''=\frac{g(x,y,y\prime)}{f(x,y,y\prime)}$$

where, $f(x,y,y')$ is a quadratic polynomial in y' and $g(x,y,y')$ is a polynomial in y' of degree atmost five.

In order to find the concrete form, we treat the case of local coordinate system, associated Riemannian space is Euclidean with orthonormal coordinate system. Then $a=1$ and $a_x=a_y=0$. If we fill b_1 and b_2 such that $b_1=b_x, b_2=b_y$ for scalar b then $b_{1y}-b_{2x}=0$. Therefore Eq. (5.3) reduces to

$$y''=\frac{(-\alpha)\left(1+(y\prime)^2\right)\left(b_x y\prime-b_y\right)b_{0;0}}{(\alpha^2+\beta^2)\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)+\alpha(b_1 y\prime-b_2)^2} \tag{5.5}$$

where, $b_{0;0}=\left(b_{1x}+b_{1y}y'\right)+\left(b_{2x}+b_{2y}y'\right)y'$. Thus, we have the following:

Corollary 5.2. *Let F^2 be a two-dimensional Finsler space with a special (α,β)- metric $L=\alpha+\sqrt{\alpha^2+\beta^2}$. If refer to an orthonormal coordinate system (x,y) with respect to α and $b_1=\frac{\partial b}{\partial x}, b_2=\frac{\partial b}{\partial y}$ for a scalar b, then the differential space of geodesic $y=y(x)$ of F^2 is given by (5.5).*

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